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GEOMETRY.

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THE ELEMENTS

OF

DESCRIPTIVE GEOMETRY;

CHIEFLY INTENDED

FOR STUDENTS IN ENGINEERING.

BY THE

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PREFACE.

The present work is intended for those students who are occupied in graphically representing the forms of bodies, and the delineations of machines. To such a class the advantage of having general methods by which the position of points, lines, and surfaces, may be determined with exactness and precision, is very obvious. Descriptive Geometry supplies this want. Invented by the genius of Monge, and pursued with ardour and success by the most eminent French Geometricians, it is now taught in almost all the universities and in the principal schools of the continent. In England it was unknown, as a branch of instruction, until lectures were given upon it by Mr. Bradley, in the Engineering Department of King's College; and the present work has been undertaken to supply the students with a text book, that by it they might the more profitably attend to what they heard in the lecture room: and as an elementary book was necessary for beginners, it has been thought expedient to place before the students, in an English dress, one which has stood the test of experience. The treatise on Descriptive Geometry, by Mr. Lefebure de Fourcy has, therefore, been selected, and the following pages are, for the most part, translated from it.

And now a few words on the subject itself. It is obvious that, in the progress of the arts, there is a constant necessity of transmitting to others the knowledge that is possessed of the forms of bodies; either for the purpose of exhibiting the geometrical relations that have been discovered, or of guiding the workman engaged in the construction of objects whose dimensions are given. To effect this, two methods may be used; 1°, a Model; 2°, a Drawing or Plan. The first method is commonly impracticable; not indeed of itself, but from the difficulty and expense of execution, and the time which would be spent in making the model. To the second method, the same objections do not apply with the same force as to the first, but another and peculiar difficulty belongs to it, which it is the special object of Descriptive Geometry to remove. For, the bodies to which reference has been made, the buildings which the workman has to erect, the machines, parts of which the drawings represent, are in different planes, while the sheet of paper which contains their graphical descriptions is in one plane. Means, therefore, must be found, by which the points in space may be referred to a single plane, and which, in every problem, shall express with precision the given and required quantities. It is this exactness which constitutes the important difference between ordinary and Descriptive Geometry. In the former, the figures, often drawn vaguely, serve to guide the mind in the series of reasonings necessary to establish the truth of a theorem or the construction of a problem. In the latter, the steadiness of the hand and eye is also requisite : one line carelessly or inaccurately drawn, vitiates the whole result.

It has been mentioned, that Descriptive Geometry owes its origin to Monge. He first conceived, that bodies might be represented by figures drawn in a single plane, by the aid of projections on two different planes, commonly perpendicular to each other, and which we may consider as respectively horizontal and vertical. One of these, as, for instance, the vertical, is supposed to turn round their common intersection until it coincides with the prolongation of the other. The vertical projections are thus reduced to the horizontal plane, and their actual distances from the line of intersection always preserved. Considering points and lines to be determined by their projections, and surfaces by the projections of their generatrices, Monge ascertained that the various problems of the geometry of three dimensions, which had been solved before his time by a variety of processes, could be reduced to a small number of abstract problems, and formed simple and entirely general methods for their graphical solution. For the development and full application of these principles, the reader is referred to the works of this great Geometer, and to those of the other French mathematicians, who have enriched the subject with a great variety of problems.

It remains for me to express my acknowledgments to Mr. Bradley, of King's College, for the kind assistance he afforded me, during the progress of this work. Without his help, I should not have ventured to undertake it: he carefully revised all that I had written, and when I was unable to give my attention to it, prepared a considerable portion of the book for the press.

T. G. HALL.

King's College, London,
October, 1841.

INTRODUCTION.

THE PLANE AND STRAIGHT LINE.

DEFINITIONS.

(1) A superficies is that which has only length and breadth.

(2) A solid is that which has length, breadth, and thickness.

(3) That which bounds a solid is a superficies.

(4) A plane superficies, or a plane, is that in which any two points being taken, the straight line between them lies wholly in that superficies.

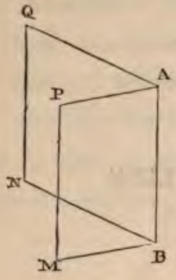
(5) A straight line is said to be perpendicular, or at right angles to a plane, when it is perpendicular to every straight line meeting it in that plane.

(6) The inclination of a line to a plane is the plane angle contained by that straight line, and another drawn from the point in which the first line meets the plane, through the point in which a perpendicular from any point of the first line above the plane meets the same plane.

(7) The inclination of a plane to a plane is the plane angle contained by two straight lines drawn from any the same point of their common section at right angles to it, one in one plane, and the other in the other plane.

(8) One plane is said to be perpendicular to another plane, when the straight lines drawn in one of the planes perpendicular to the common section of the two planes are perpendicular to the other plane.

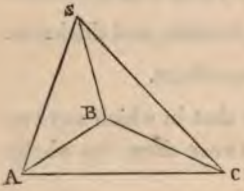
(9) Planes which do not meet one another, though produced, are said to be parallel.



(10) The angle between two planes is called the dihedral angle.

Thus, Q A B M is the dihedral angle of the two planes, Q A B N, P A B M: A B is called the edge, and the planes containing the angles are called faces.

(11) A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.



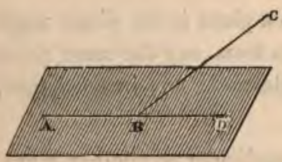
Thus, the solid angle at S is contained by the three plane angles, A S B, A S C, B S C; the planes A S B, A S C, B S C, are called faces; and S the vertex of the solid angle.

(12) A solid angle is trihedral, tetrahedral, pentahedral, &c., according as it is contained by three, four, or five, &c., faces.

Proposition I.

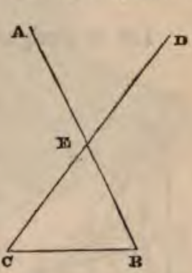
One part of a straight line cannot be in a plane, and the other part above it.

If possible, let A B, part of the straight line A B C, be in the plane, and the part B C above it. Then ∵ the straight line A B is in the plane, it can be produced in that plane: let it be produced to D; and let any plane pass through the straight line A D, and revolve till it pass through the point C; then ∵ the points B and C are in this plane; B C is in it; ∴ two straight lines A B C, A B D, in the same plane, have a common segment A B, which is impossible.



Proposition II.

Two straight lines which cut one another are in the same plane, and three straight lines which meet one another are in one plane.



Let the two straight lines AB, CD, intersect at E; they shall be in the same plane; and EC, CB, BE, which meet one another, shall be in one plane.

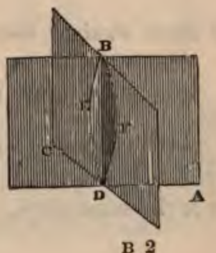
Let any plane pass through the straight line EB, and let the plane on it produced revolve round EB, till it pass through the point C.

Then ∵ E and C are in this plane, the straight line EC is also in it; similarly BC is in it; and EB is in it by hypothesis ∴ EC, CB, BE, are in one plane; but in the plane in which EC, EB, are, in the same are CD, AB: ∴ AB and CD are in one plane.

Cor. Hence a triangle EBC, or three points EBC, determine a plane.

Proposition III.

If two planes cut one another, their common section is a straight line.

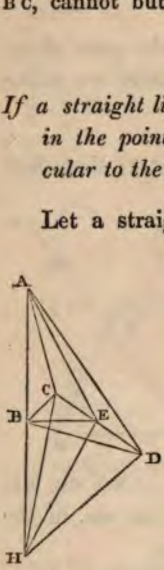


Let two planes, A B, C D, cut one another, and let the line D B be their common section: D B shall be a straight line.

For if not, from D to B, draw in the plane A B, the straight line D E B, and in the plane B C, the straight line D F B: then the two straight lines D E B, D F B, have the same extremities and ∴ include a space betwixt them, which is impossible: $\therefore$ BD, the common section of the planes AB, BC, cannot but be a straight line.

PROPOSITION IV.

If a straight line be perpendicular to each of two straight lines in the point of their intersection, it shall also be perpendicular to the plane passing through them.



Let a straight line AB, be $\perp$ to two straight lines BC, BD, meeting at the point B, it is also $\perp$ to MN, the plane in which they are.

To prove this proposition, it must be shown that AB is $\perp$ to any straight line BE, drawn from B, in the plane MN. Produce AB to H; make $BH = AB$: draw any straight line CED, cutting BC, BE, BD, in C, E, D. Join AC, AE, AD; HD, HE, HC.

Then $\because AB = HB$, and $\angle ABC = \angle HBC$, and BC is common to the $\triangle^s$ ABC, HBC: $\therefore AC = HC$: and similarly $AD = HD$. Again in $\triangle^s$ ACD, HCD, all the sides are $=$; $\therefore \angle ACE = \angle HCE$ $\therefore$ in $\triangle^s$ ACE, HCE; $AC = HC$, CE is common, and $\angle ACE = \angle HCE$ $\therefore AE = HE$.

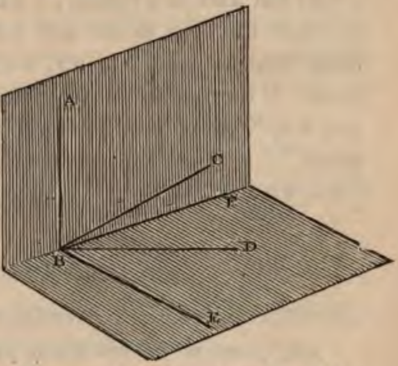
$\therefore$ in $\triangle^s$ ABE, HBE, $AB = HB$, BE is common, and $AE = HE$ $\therefore \angle ABE = \angle HBE$, and they are adjacent angles. $\therefore \angle ABE$ is a right angle, and consequently AB is $\perp$ to the plane MN.

PROPOSITION V.

If three straight lines meet in one point, and a straight line be perpendicular to each of them in that point, these three straight lines are in the same plane.

Let the straight line AB be $\perp$ to each of the straight lines BC, BD, BE, in B the point where they meet: BC, BD, BE, shall be in the same plane.

If not, if possible let BD, BE, be in one plane, and let BC be above it: and let a plane pass through AB, BC, the common section of which, with the plane in which BD and BE are, is a straight line; let this be BF.



$\therefore$ AB, BC, BF, are all in one plane: viz., that through AB, BC: and $\because$ AB is $\perp$ to both BD and BE, it is $\perp$ to the plane through BD and BE: and is $\therefore$ $\perp$ to BF, which is in that plane. (Prop. IV.)

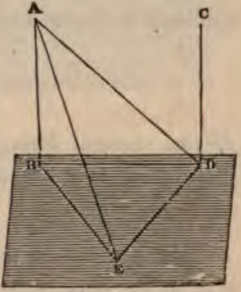
$\therefore$ $\angle$ ABF is a right $\angle$, but $\angle$ ABC is a right $\angle$ by hypothesis. $\therefore$ $\angle$ ABF $=$ $\angle$ ABC, and they are both in the same plane, which is impossible.

$\therefore$ BC is not above the plane in which are BD and BE; similarly it is not below it: $\therefore$ BC, BD, BE, are in the same plane.

Proposition VI.

If two straight lines be perpendicular to a plane, they shall be parallel to each other.

Let AB, CD, be $\perp$ to the same plane; AB shall be parallel to CD.



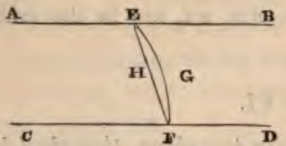
Let them meet the plane in BD: join BD: draw DE $\perp$ to BD, and in the same plane with it: make DE $=$ AB: join BE, AE, AD. Then $\because$ AB is $\perp$ to the plane in which BD, BE, are, each of the $\angle^s$, ABD, ABE, is a right $\angle$: $\therefore$ also $\angle$ CDB, and $\angle$ CDE are right $\angle^s$; and $\because$ AB $=$ DE, and BD is common, and $\angle$ ABD $=$ $\angle$ BDE; $\therefore$ AD $=$ BE.

Again, ∵ AB = DE; BE = AD, ∴ AB, BE = AD, DE, each to each, and AE is common to the $\triangle^s$ ABE, ADE. ∴ ∠ ABE = ∠ ADE; but ABE is a right ∠, ∴ ADE is a right ∠, and ED is ⊥ DA: but ED is ⊥ to BD and DC: ∴ ED is ⊥ to each of the three straight lines, DB, DA, DC, in the point in which they meet. ∴ they are in the same plane; but AB is in the plane of BD, DA, ∵ three straight lines which meet are in the same plane, ∴ AB, BD, DC, are in one plane; and each of the $\angle^s$ ABD, BDC, is a right ∠ ∴ AB is parallel to CD.

Proposition VII.

If two straight lines be parallel, the straight line drawn from any point in the one to any point in the other is in the same plane with the parallels.

Let AB, CD, be parallel straight lines: take any point E in the one, and any point F in the other: the straight line joining E and F shall be in the plane of the parallels.



For if not, let it, if possible, be above the plane, as EGF: and in the plane ABCD of the parallels: draw the straight line EHF from E to F: and ∵ EGF is also a straight line, the two lines EGF, EHF, include a space, which is impossible. Therefore the straight line joining E and F is not above the plane ABCD; similarly it cannot be below it: it is ∴ in the plane.

Proposition VIII.

If two straight lines be parallel, and one of them be perpendicular to a plane: the other shall be also perpendicular to the same plane.

Let AB, CD, be the two parallels, and AB ⊥ to a plane; CD is ⊥ to the same plane.

Let AB, CD, meet the plane in B and D: join BD: ∴ AB, CD, BD, are in one plane. (Prop. VII.)

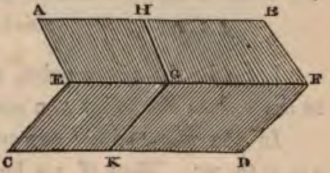
In the plane to which AB is ⊥, draw DE ⊥ BD and = AB: join BE, AE, AD: then ∵ AB is ⊥ to the plane, each of the ∠^{s} ABE, ABD, is a right ∠; and ∵ BD meets the parallels AB, CD; ∴ ∠ ABD + ∠ CDB = two right ∠^{s}; but ∠ ABD is a right ∠, ∴ ∠ CDB is a right ∠: ∴ CD is ⊥ BD.

And ∵ AB = DE, and BD is common: and ∠ ABD = ∠ EDB, ∴ AD = EB. Again, ∵ AB = DE, BE = AD, and AE is common. ∴ ∠ ABE = ∠ EDA: but ∠ ABE is a right ∠, ∴ EDA is so also, and ED is ⊥ DA: but ED is ⊥ BD, ∴ to the plane through BD, DA, but DC is in that plane; ∴ ∠ EDC is a right ∠, or CD is ⊥ DE, but CD is ⊥ BD, ∴ to the plane through BD, DE, *i.e.*, to the same plane to which AB is ⊥.

Proposition IX.

Two straight lines, which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another.

Let AB, CD, be each parallel to EF, and not in the same plane with it, AB shall be parallel to CD.



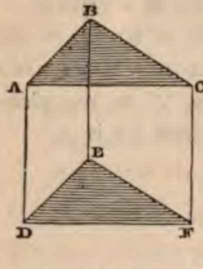
In EF take any point G: from G, draw in the plane through EF, AB, GH ⊥ to EF: and in the plane through EF, CD, draw GK ⊥ EF.

Then ∵ EF is ⊥ to both GH and GK, it is ⊥ to the plane HGK: and EF is parallel to AB: ∴ AB is ⊥ to HGK. Similarly CD is ⊥ to HGK, ∴ AB and CD being both ⊥ to the same plane are parallel. (Prop. VI.)

PROPOSITION X.

If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the former two, the former two and the other two shall contain equal angles.

Let AB, BC, which meet in B, be respectively parallel to DE, EF, meeting in E, but not in the same plane as AB, BC; the ∠ ABC = ∠ DEF.



Take BA, BC, ED, DF, all =; join AD, CF, BE, AC, DF: then ∵ BA is = and parallel to DE, ∴ AD is = and parallel to BE: for the same reason CF is = and parallel to BE: ∴ AD and CF being each of them = and parallel to BE, ∴ they are = and parallel to each other: ∴ AC and DF, which join their extremities towards the same parts, are also = and parallel. But ∵ AB, BC = DE, EF, each to each, and AC = DF, ∴ ∠ ABC = ∠ DEF.

PROPOSITION XI.

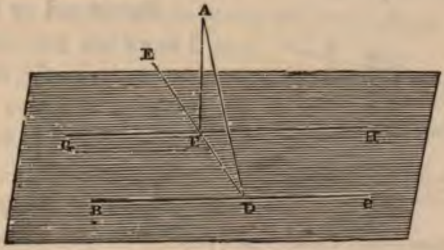
To draw a straight line perpendicular to a plane from a given point above it.

Let A be the given point, BH the given plane; it is required to draw from the point A, a straight line ⊥ to BH.

In the plane draw any straight line BC, and from A draw AD ⊥ to BC. Then if AD be also ⊥ to BH, AD is the ⊥ required: but if not, in BH, draw DE ⊥ BC: and from A draw AF ⊥ DE: AF is ⊥ to the plane BH.

Through F draw GH parallel to BC.

Then ∵ BC is ⊥ both to ED and DA, it is ⊥ to the plane through them; and GH is parallel to BC. ∴ GH is also ⊥ to the plane EDA and ∴ GH is ⊥ to AF: and conversely AF is ⊥ to GH: and AF is ⊥ to DE. ∴ AF is ⊥ to the plane through ED and GH, *i.e.*, to the plane BH.

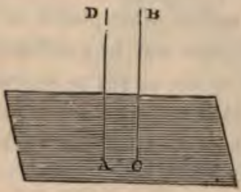


COR. Hence it follows that if AF be a ⊥ to a plane, and FD a ⊥ to any line BC in that plane, and AD be joined, that AD is ⊥ to BC.

PROPOSITION XII.

From a given point in a plane to draw a perpendicular to the plane.

Let A be the given point in the plane: take any point B above the plane, from which draw BC ⊥ to the plane: and then from A draw AD parallel to CB: AD is the ⊥ required. For BC being ⊥ to the plane, and AD parallel to BC, AD is also ⊥ to the plane.

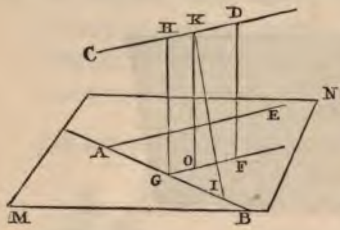


PROPOSITION XIII.

To find the perpendicular distance of two lines not in the same plane.

Let AB, CD, be the two lines. Through any point A in AB, draw AE ∥ to CD, and let MN be the plane passing through AB and AE.

From any point D in CD draw DF ⊥ to MN: and from F in MN, draw FG || to AE: from G draw GH || to FD: GH is the perpendicular required.



For GF and CD being both || to AE are || to each other; ∴ GF is in the same plane with CD and DF.. And thus GH, which is || to FD, is in the same plane, and ∴ H is in CD.

Also, DF being ⊥ to the plane MN, GH which is || to it is also ⊥ MN: and ∴ to AB and GF.

But GF is || to CD ∴ GH is also ⊥ to CD.

Also, GH is the shortest distance.

For if not, let KI be the shortest distance between AB and CD; then draw KO || to GH and FD. ∴ KO is ⊥ to MN; ∴ KI is > KO, but KO = GH, ∴ KI is > GH, ∴. GH is the shortest distance between AB and CD.

In the last three problems, lines are assumed to be drawn from points || or ⊥ to other lines. This assumption, both in this and in other places, is necessary in order that the properties of figures may be exhibited; but the student must be reminded that he has as yet no practical method of drawing those lines, and that the effectual solution of these problems is one of the special objects of this work.

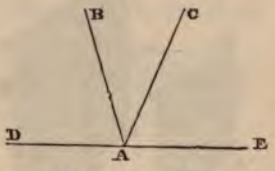
PROPOSITION XIV.

From the same point in a given plane there cannot be two straight lines perpendicular to the plane, upon the same side of it; and there can be only one perpendicular to a plane from a point above the plane.

If possible, let AB, AC, be both ⊥ to a plane, from the same point A, and upon the same side of the plane.

Let a plane pass through BA, AC, and let DAE be the common section of it with the given plane: $\therefore$ AB, AC, DAE, are in one plane.

Then $\because$ CA is $\perp$ to the plane, and meets DAE, $\therefore$ $\angle$ CAE is a right $\angle$; for the same reason BAE is a right $\angle$.



$\therefore$ $\angle$ CAE $=$ $\angle$ BAE, the less to the greater, which is impossible.

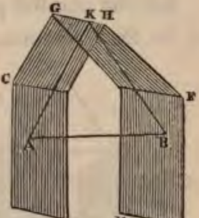
Also, from the same point above a plane there can be drawn but one perpendicular to that plane; for if there could be two they would be $\parallel$, which is absurd.

PROPOSITION XV.

Planes to which the same straight line is perpendicular are parallel to one another.

Let AB be $\perp$ to each of the planes CD, EF; these planes are $\parallel$ to each other.

If not, they will meet if produced: let them meet; their common section is a straight line GH, in which take any point K, and join AK, BK.



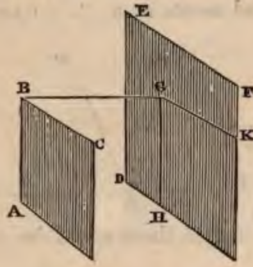
Then $\because$ AB is $\perp$ EF it is $\perp$ to BK $\therefore$ $\angle$ ABK is a right $\angle$; for the same reason KAB is a right $\angle$; $\therefore$ two $\angle^s$ of the $\triangle$ ABK are $=$ two right $\angle^s$, which is impossible; $\therefore$ the planes CD, EF, though produced, do not meet; therefore they are $\parallel$.

PROPOSITION XVI.

If two straight lines meeting one another be parallel to two other straight lines which meet one another, but are not in the same plane with the first two; the plane which passes through these is parallel to the plane through the others.

Let AB, BC, meeting in B, be $\parallel$ to DE, EF, meeting in E, but which are not in the same plane with AB, BC: the planes through

AB, BC, and DE, EF, are ∥. From B, draw BG ⊥ to plane through ED, DF, through G, where BG meets the plane; draw GH ∥ to ED, and GK ∥ to EF.



And ∵ BG is ⊥ to plane DF, each of the $\angle^{s}$ BGH, BGK, is a right ∠; and ∵ BA is ∥ to GH, (for both are ∥ to DE,) the ∠ GBA + ∠ BGH = two right $\angle^{s}$.

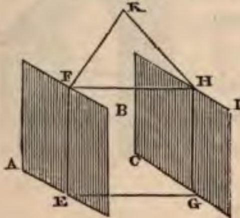
But BGH = a right ∠, ∴ ∠ GBA = right ∠, ∴ GB is ⊥ to BA. For the same reason GB is ⊥ BC, and ∴ GB is ⊥ to the plane through AB, BC; and it is ⊥ to plane through DE, EF.

∴ BG is at once ⊥ to the plane through AB, BC, and the plane through DE, EF; hence these planes are parallel.

Proposition XVII.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the parallel planes, AB, CD, be cut by EFHG, and let EF, HG, be their common sections: EF shall be parallel to GH.



If not, EF, GH, will meet if produced either on the side of FH or EG. First let them be produced on the side of FH, and meet in the point K.

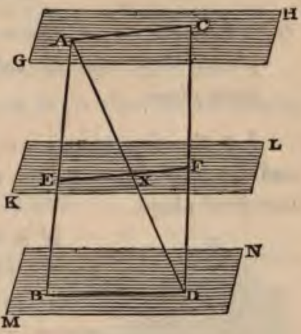
Hence, ∵ EFK is in the plane AB, every point of EFK is in that plane, ∴ K is a point in the plane AB. For the same reason, K is a point in the plane CD; ∴ the planes AB, CD, meet if produced, which is contrary to the hypothesis.

∴ EF and GH do not meet when produced on the side of FH. In the same manner it may be shown that they do not meet when produced on the side EF: wherefore EF and HG are parallel.

PROPOSITION XVIII.

If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

Let AB, CD, be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D; then AE : EB :: CF : FD.



Join AC, BD, AD; let AD meet KL in X; join EX, XF. Then ∵ the parallel planes KL, MN, are cut by EXDB, ∴ EX is parallel to BD; and ∵ GH, KL, are cut by the plane AXFC, the common sections AC, XF, are parallel. Then ∵ EX is parallel to BD in △ ABD, and ∵ XF is parallel to AC in △ ADC,

∴ AE : EB :: AX : XD.

And AX : XD :: CF : FD.

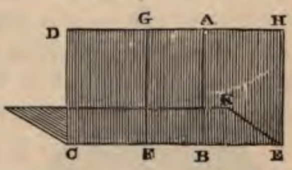
∴ AE : EB :: CF : FD.

PROPOSITION XIX.

If a straight line be perpendicular to a plane, every plane passing through it shall be perpendicular to that plane.

Let AB be ⊥ to plane CK; every plane through AB shall be ⊥ to CK.

Let plane DE pass through AB; and let CE be the common section of DE and CK. Take any point F in CE, draw FG in DE, ⊥ to CE; then ∵ AB is ⊥ to CK, ∠ ABF is a right ∠, but GFB is a right ∠, ∴ AB and FG are ∥, and AB is ⊥ to the plane CK, ∴ FG is ⊥ to the same plane. But one plane is ⊥ to another plane, when the straight lines drawn in one of the planes, ⊥ to their

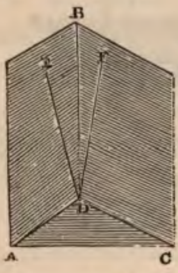


common section, are also $\perp$ to the other plane. And any line FG, in the plane DE, drawn $\perp$ to the line CE, is also $\perp$ to CK. $\therefore$ the plane DE is $\perp$ to CK. And the same may be proved of any other plane passing through AB.

PROPOSITION XX.

If two planes which intersect be each perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Let the two planes AB, BC, be each $\perp$ to the third plane, and let BD be the common section of AB, BC; BD shall be $\perp$ to the third plane.



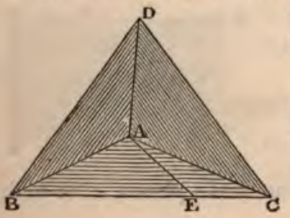
If it be not, from D in AB draw DE $\perp$ AD, the common section of AB with the third plane; in BC draw DF $\perp$ CD, the common section of BC with the same plane.

Then $\because$ AB is $\perp$ to plane ADC, and DE is $\perp$ to AD, $\therefore$ DE is $\perp$ to ADC; similarly, DF is $\perp$ to ADC; $\therefore$ from the same point D, two $\perp^s$, DE, DF, to the same plane ADC, have been drawn, which is impossible; $\therefore$ the only line that can be drawn from D $\perp$ to ADC is BD, the common section of AB and BC.

PROPOSITION XXI.

If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let the solid $\angle$ A, be contained by the three plane $\angle^s$, BAC, CAD, DAB; any two of them shall be greater than the third.



If the three angles be all $=$, the proposition is evident. If not, let BAC be not less than either of the other two, and $>$ DAB. At the point A, in the plane through BA, AC, make $\angle BAE = \angle DAB$. Make $AE = AD$; through E draw BEC, cutting AB, AC, in B and C, and join DB, DC.

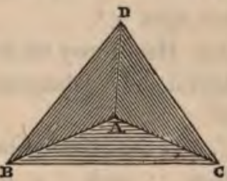
And $\because DA = AE$, and AB is common, and $\angle DAB = \angle EAB$, $\therefore DB = EB$; but $\because$ BD, DC, are $>$ BC and $BD = BE$, $\therefore$ DC is $>$ EC; and $\because DA = AE$ and AC is common, also $DC > EC$, $\therefore \angle DAC$ is $> \angle EAC$; and $\angle DAB = \angle BAE$, $\therefore \angle DAC + \angle DAB$ is $> \angle BAC$.

Also, $\angle BAC$ is not less than either of the other two; $\therefore \angle BAC +$ either of the other two, must be $>$ than the remaining angle.

Proposition XXII.

Every solid angle is contained by plane angles, which together are less than four right angles.

First, let the solid $\angle$ A be contained by the three plane $\angle^s$, BAC, CAD, DAB; take in each of the lines AB, AC, AD, any points B, C, D; join BC, CD, DB. Then $\because$ the $\angle^s$ CBA, ABD, DBC, make the solid $\angle$ B, $\therefore \angle CBA + \angle ABD$ are $> \angle DBC$; similarly $\angle BCA + \angle ACD$, are $> \angle DCB$; and $\angle CDA + \angle ADB$, are $>$ BDC. $\therefore \angle^s$ CBA, ABD, BCA, ACD, CDA, ADB, are $> \angle^s$ DBC, BCD, BDC; but $\angle DBC + \angle DCB + \angle BDC =$ two right $\angle^s$.



$\therefore \angle CBA + \angle ABD + \angle BCA + \angle ACD + \angle CDA + \angle ADB$ are $>$ two right $\angle^s$; and $\because$ the three $\angle^s$ of each of the $\triangle^s$ ABC, ACD, ADB, are $=$ two right $\angle^s$, $\therefore$ the nine $\angle^s$ of these $\triangle^s$, viz., the $\angle^s$ CBA, BAC, ACB, ACD, ADC, DAC, ADB, ABD, BAD, are $=$ six right $\angle^s$; and of these the $\angle^s$ CBA, ABD, BCA, ACD, ADC, ADB, are $>$ two right $\angle^s$. $\therefore$ the three remaining $\angle^s$, BAC, BAD, DAC, are $<$ four right $\angle^s$.

Next, let the solid $\angle$ at A be contained by any number of plane $\angle^s$, BAC, CAD, DAE, EAF, FAB; these together shall be less than four right $\angle^s$.

Let the planes in which the $\angle^s$ are, be cut by a plane, and let BC, CD, DE, EF, FB, be its common sections with the planes. Then $\because$ the solid $\angle$ B is contained by three plane $\angle^s$, CBA, ABF, FBC, of which any two are $>$ the third, $\angle$CBA $+$ $\angle$ABF are $>$ $\angle$ FBC; for the same reason, the two plane $\angle^s$ at each of the points C, D, E, F, viz., those $\angle^s$ which are at the bases of the $\triangle^s$ having a common vertex A, are greater than the third $\angle$ at the same point, which is one of the angles of the polygon, BCDEF; $\therefore$ all the $\angle^s$, at the bases of the $\triangle^s$, are together greater than all the $\angle^s$ of the polygon; and $\because$ all the $\angle^s$ of the $\triangle^s$ $=$ twice as many right $\angle^s$ as there are $\triangle^s$, *i.e.*, as there are sides in the polygon BCDEF; and $\because$ all the $\angle^s$ of the polygon $+$ four right $\angle^s$ $=$ twice as many right $\angle^s$ as there are sides in the polygon; $\therefore$ all the $\angle^s$ of the $\triangle^s$ are $=$ all the $\angle^s$ of the polygon $+$ four right $\angle^s$; but the $\angle^s$ at the bases of the $\triangle^s$ are $>$ $\angle^s$ of the polygon; $\therefore$ the remaining $\angle^s$ of the $\triangle^s$, which are those at the vertex, and contain the solid at A, are $<$ four right $\angle^s$.

COR. Hence may we find how many of the plane $\angle^s$ of an equilateral and equiangular polygon be taken to form a solid angle.

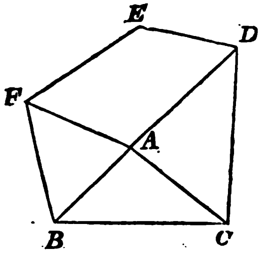
1°. Let the plane $\angle$ be that of an equilateral $\triangle$.

Then $\because$ each $\angle = 60°$; and $\because$ $3 \times 60 = 180$, $4 \times 60 = 240$, $5 \times 60 = 300$: a solid $\angle$ may be formed of 3, 4, 5 equal $\angle^s$ of an equilateral $\triangle$; but not more, $\because$ $6 \times 60 = 360 =$ four right $\angle^s$. The solids so formed are respectively called the Tetrahedron, the Octahedron, the Icosahedron.

2°. Let the equilateral polygon be a square; then each plane $\angle = 90°$; and $\because$ $3 \times 90 = 270 < 360$, a solid angle with three right $\angle^s$ may be formed, but not more, $\because$ 4×90 is not <360. This solid is the cube.

3°. Let the polygon be the pentagon; the interior $\angle$ of which $= 108°$; and $\because$ $3 \times 108 = 324 < 360$. A solid $\angle$, containing three of these ∠s, may be formed. A regular solid, which has this kind of ∠, is called the Dodecahedron.

As the interior ∠ of a hexagon = 120°, no solid ∠ can be formed of the plane ∠s of a regular hexagon; and the same may be said of the ∠s of a polygon containing more than six sides.



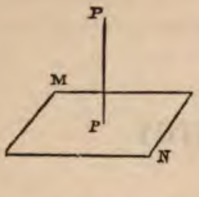
DESCRIPTIVE GEOMETRY.

1. The object of Descriptive Geometry is the invention of methods by which we may represent upon a plane, having only two dimensions, length and breadth, the form and position of a body which possesses three dimensions, length, breadth, and height.

This definition of the chief object of descriptive geometry includes its application to points and lines situated above or below the plane, to which they are to be referred; and thus gives us means of drawing such lines and points with an accuracy unnecessary and unknown to common geometry. For the results obtained in plane geometry do not depend upon the faithfulness with which the figures are drawn, but upon the accuracy with which the reasoning is conducted. But even in these problems we often make assumptions with regard to the mechanical drawing of lines, which we should find it extremely difficult to execute. Thus in Proposition XI. of the Introduction we say, draw $AD \perp BC$; but the practical difficulty of drawing AD is as great as that of drawing AH, the object of the problem.

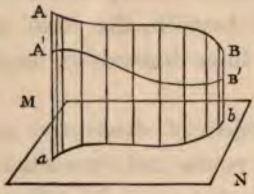
The means by which descriptive geometry attains its object is the method of Projections, which we now proceed to explain.

2. The projection of a point upon a plane is the foot of the perpendicular let fall from the point upon the plane.



Thus, if Pp be drawn from P, perdicular to the plane MN, p is the projection of P upon the plane MN. And MN is called the *plane of projection*. It is also evident that p is the projection of every point of the line Pp.

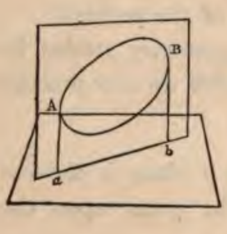
3. The projection of a line upon a plane is the line formed by the projections of every point of the former line.



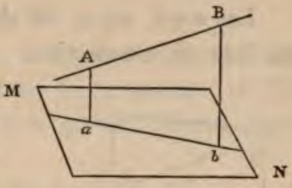
From the different points of the curve AB, let fall perpendiculars on the plane MN; the curve line ab, formed by the feet of these perpendiculars, is the projection of the curve AB. The perpendiculars so drawn will form a surface, called, from the parallelism of the lines which may be supposed to generate it, a cylindrical surface; and for this reason it is often known by the name of the *projecting cylinder*.

The projection ab may be considered as the intersection of the projecting cylinder with the plane MN; and it is evident that every curve, as A′B′ traced on this cylinder, will have ab for its projection.

4. When AB is a plane curve, and in a plane perpendicular to the plane of projection, the projecting cylinder coincides with the plane of the curve; and the projection of the curve AB is the straight line ab: but when the curve is in a plane parallel to the plane of projection, it is manifest that the projection is an equal curve.



5. The projection of a straight line AB is a line; for the perpendiculars let fall from the different points of AB are all in the same plane.



And since two points determine a line, we have to draw the projection of AB, merely to find the projections a, b, of any two points A, B; then join a, b, and the line ab, or that line produced, will be the projection required.

A plane drawn through AB ⊥ to MN is called the *projecting plane* of the line AB.

Hence to find the projecting plane of AB, we draw from any point A, Aa, perpendicular to the plane MN; and suppose a plane to pass through the two lines AB and Aa, it will be the plane required. If AB be perpendicular to MN, its projection is the point where AB or AB produced meets MN.

6. A point is determined in space, when its projections, upon two planes which intersect, are given.

For the point should be found upon each of the perpendiculars drawn to the planes from the given projections; and these perpendiculars can only intersect in one point, which is the point required.

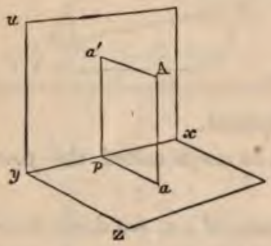
The two planes to which reference is here made, and which are generally at right angles to each other, are called the planes of projection, or co-ordinate planes.

The following proposition will determine when two points in the co-ordinate planes are the projections of the same point.

7. When two points, situated in two co-ordinate planes, are the projections of the same point in space, the per-

pendiculars drawn from these points upon the line of intersection of the planes, must meet at the same point in that line.

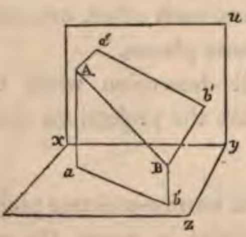
Let xyz, xyu, be the two co-ordinate planes; xy the line of intersection. From any point A, draw the perpendiculars, Aa, Aa': and through Aa, Aa', draw the plane $Aa'pa$, cutting xy in p, and the two planes in ap and $a'p$; then the line xy shall be $\perp$ to the plane $Aapa'$ and $\therefore$ to ap and $a'p$; and thus the $\perp^s$, from the projections a and a' of the point A, meet in the same point.

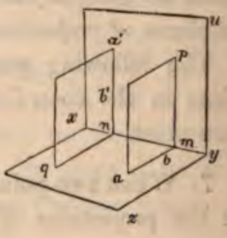


Next, the points a and a' are always the projection of the same point in space.

The line xy being $\perp$ to ap and $a'p$, is also $\perp$ to the plane apa'; and $\therefore$ the planes xyz, xyu, are $\perp$ to plane apa'; $\therefore$ conversely apa' is $\perp$ to xyz, and xyu. $\therefore$ if from a and a' perpendiculars be drawn to the planes xyz, xyu, these lines will be in the plane apa'; whence it is evident that they ought to intersect in the point A, of which the projections are a and a'.

8. A line is known, when its projections upon two planes which intersect are given.

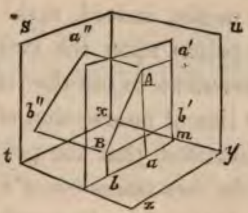




Let ab, $a'b'$, be the projections of a line; through ab conceive a plane $\perp$ to xyz, and through $a'b'$, a plane $\perp$ to xyu; the intersection AB of these two planes is the line of which the projections are ab and $a'b'$.

If the two lines ab, $a'b'$ are $\perp$ to xy, but intersect it in different points m, n, they cannot be the projections of the same line, since they are obviously in parallel planes, pma and qna'.

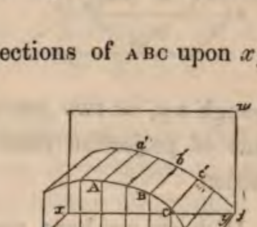
But if the projections ab, $a'b'$, be both $\perp$ to xy, and meet xy in the same point m, then the plane ama' of these lines is $\perp$ both to xyz, and xyu, so that every line situated in ama', will have the same projections ab and $a'b'$.



In this case we must introduce a third plane txs, which does not pass through xy, and the line may be determined from its projection $a''b''$ upon txs.

9. The projections of any curve upon two intersecting planes will determine the curve.

For abc, $a'b'c'$, being the projections of ABC upon xyz and xyu, suppose perpendiculars to these planes to be drawn from the different points of abc, $a'b'c'$; these will form two cylindrical surfaces, $accA$, $a'c'cA$, both of which will contain the curve ABC, and which will determine that curve by their intersection.



If the cylindrical surfaces do not intersect, the projections do not belong to the same curve.

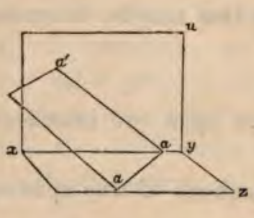
If ABC be a plane curve, and its plane be $\perp$ to xy, the two projections are merely the intersections of the plane through ABC, and the curve remains undetermined.

Def. The intersections of a plane with the co-ordinate planes are called the traces of the plane.

10. When the traces of a plane are given, the plane itself is given.

We have already seen (Proposition II. of the Introduction), that two lines which intersect determine a plane; we might, indeed, make use of the projections of three of its points upon the two co-ordinate planes, but it is more convenient to use the traces. In general, the plane meets the line of intersection of the two co-ordinate planes, and it is manifest that the point of intersection a, must belong to the two traces aa, $a'a$.

If the plane be parallel to xy, its traces are also parallel to xy. If it be $\perp$ to xy, its traces are also $\perp$ to xy. If it be parallel to one of the two planes, its trace upon the other will be parallel to xy, and will be sufficient to determine the plane; but if the plane pass through xy, its two traces coincide with xy, and a third plane of projection must be introduced to determine its position.



Hitherto we have made no hypothesis with regard to the angle between the two planes of projection. But to render the constructions the more simple, we shall hereafter consider the two planes to be at right angles to each other, and that one of them, xyz, is *horizontal*, and the other, xyu, *vertical*. Their intersection, xy, is called the *ground line*.

If a line be parallel to the horizontal plane, it is said to be horizontal; and to be vertical when it is perpendicular to the horizontal plane. But a line is not necessarily vertical when it is parallel to the vertical plane.

Each projection of a line, or each trace of a plane, takes its name from the plane which contains it; thus the projections and traces situated in the vertical plane, are the vertical projections and traces.

In consequence of the planes of projection being at right angles to each other, we deduce:

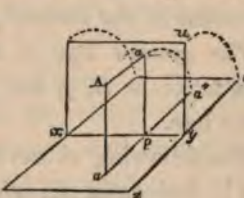
1°. If a point or a line be in one of the two planes, its projection upon the other plane will be a point in the ground line.

2°. If a line be parallel to one of the two planes of projection, its projection on the other plane will be parallel to the ground line.

3°. If a plane be perpendicular to one of the planes of projection, its trace upon the other plane will be perpendicular to the ground line.

4°. The perpendiculars to the ground line, from the projections of a point, are respectively equal to the distances of that point from the planes of projection; for it is manifest that $Aa' = ap$, and $Aa = a'p$. (See fig., p. 21.)

11. In order that all the constructions may be made in one plane; the vertical plane xyu, is supposed to turn round the intersection xy, as if xy were a hinge, until it coincides with the plane xyu', which is xyu produced upon the other side of xy.



The lines in the plane of xyz undergo no change; but after the revolution round xy, the projections a and a' have a connexion which must be carefully observed and remembered.

Let a and a' be the original projections of A, before the motion of the plane xyu: we have seen that the perpendiculars from a' and a, upon xy, meet in the same point p. Now, while xyu revolves round xy, $a'p$ is still $\perp$ to xy; and when xyu coincides with xyu, $a''p$, which is the position of $a'p$, is in the same plane with ap, and $\perp$ to the same line xy, and at the same point p in xy: $\therefore$ ap and $a''p$ must be in one and the same straight line; whence we see that the *straight line joining the horizontal and vertical projections of the same point* A is perpendicular to the ground line xy; and that the part $a''p$ gives the elevation of the point A above the horizontal plane, and that ap is the distance of it from the vertical plane.

12. We now proceed to give problems relative to the straight line and the plane. The drawing that contains the construction of the problem is called the *Draught*.

The given quantities and the results of the problems are generally expressed by continuous lines, and the lines of construction by dotted lines.

And that the explanations of the drawings may be readily understood, a uniformity in the notation is very necessary. We therefore indicate points in space by capital letters, A, B, C, &c., but these rarely appear in the drawings. The small letters abc, &c., belong to the projections in the horizontal plane; the accentuated letters $a'b'c'$, &c., indicate projections in the vertical plane.

The ground line is always designed by xy, and the Greek letters α, β, γ, &c., generally indicate points situated in that line.

The following abbreviations are also used:—The point $[a, a']$, designates the original point A, which has a and a'

for its projections; the line [ab, $a'b'$], that line which has for its projections ab and $a'b'$; and the plane aaa', that of which the traces are aa and aa'.

And when a point, a line, or a plane, is said to be known, it is meant that the projections of these are known. And conversely, when a point, a line, or a plane, is to be determined, it is sufficient to find the projections of the point, those of the line, or the traces of the plane.

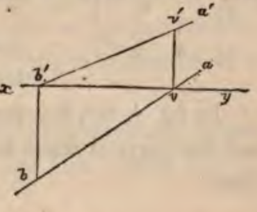
PROBLEM 1.

Given the projections of a line to find its traces, or the points where the line meets the planes of projection.

13. The straight line in space is the intersection of its projecting planes; and the point of intersection of the horizontal traces of these planes is evidently in both planes, and thus is a point of the line required: similarly, the intersection of the vertical traces of the two projecting planes is the vertical trace of the line.

Let ab and $a'b'$ be the two given projections, meeting the ground line in a and b' A vertical plane through ab will have ab for its horizontal, and vv', $\perp$ xy at v, for its vertical, trace.

A plane through $a'b'$ $\perp$ to the vertical plane will be the other projecting plane of the line, and $a'b'$ will be its vertical, and $b'b$ $\perp$ to xy in the horizontal plane its horizontal trace.



The point b, where the horizontal traces of the projecting planes intersect, and v', where the vertical traces intersect, are two points of the line in space, and, being also in the co-ordinate planes, are the traces required.

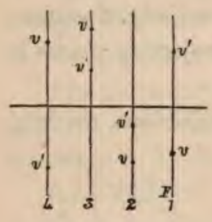
Hence, we obtain this practical rule to find the vertical trace:—Produce the horizontal projection of the line to meet the ground line; from v, the point of intersection, draw $vv' \perp xy$, meeting the vertical projection $a'b'$ in v': v' is the vertical trace; and, to find the horizontal trace, draw from b', where $a'b'$ meets xy, $b'b \perp xy$, to meet the horizontal projection ab in b: b is the horizontal trace.

14. Hence, conversely, given the traces of a line, its horizontal and vertical projections, may be found.

For, given v' and b, we have merely to draw $v'v$ and bb' $\perp$ to the ground line xy: then join vb and $v'b'$: these lines produced will be the projections required.

15. If we change the projections ab and $a'b'$, the traces v' and v can take a variety of different positions.

In fig. 1, the horizontal trace is in front of the ground line, and the vertical trace above it.



In fig. 2, the horizontal trace is still before the ground line, but the vertical trace is in the lower part of the vertical plane.

In fig. 3, the horizontal trace is behind the ground line, and the vertical trace above it.

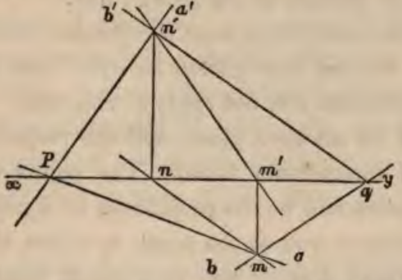
In fig. 4, the horizontal trace is behind the ground line, and the vertical trace is in the lower part of the vertical plane.

PROBLEM 2.

Given the traces of two planes, to find the projections of the common section of the planes.

16. Let m be the point of intersection of the two horizontal traces ap and bq of the planes, n' that of the vertical traces, $a'p$ and $b'q$.

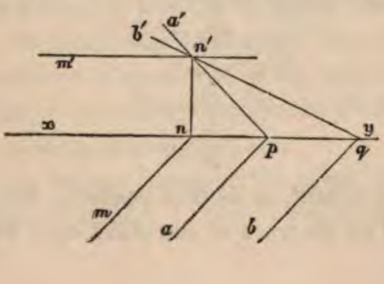
These points are common to the two planes, and the line which joins them in space is evidently the intersection of the planes. The point m being in the horizontal plane is the horizontal trace of the line; and if mm' be drawn ⊥ to xy, m' will be the vertical projection of m. In the same way, n' is the vertical trace of the line, and n the horizontal projection of n'. If, therefore, we draw the lines mn and $m'n'$, these are the projections required.



The following cases require attention:—

1°. Let one of the planes have one of its traces, the horizontal trace, for instance, perpendicular to the ground line; this plane is perpendicular to the vertical plane, and its vertical trace is the vertical projection of its intersection with the other given plane. The rest of the construction is the same as above.

2°. Suppose two of the traces, viz., the horizontal traces to be parallel; the intersection of the two planes will be

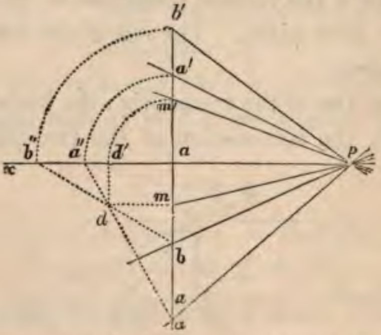


parallel to these traces, and the horizontal projection *m n*, of the intersection will be also parallel to them, and its vertical projection *m′n′* parallel to the ground line.

3°. Let the two planes meet the ground line at the same point; and let the two planes be *apa′* and *bpb′*. The general construction will not apply to this case. The planes must be cut by another plane, and the projections of the two lines of intersection found. The points where these projections meet will be the projections of a point common to the two planes; and as the point *p*, where these planes meet the ground line, is also common to them, it follows that the projections of their intersection are completely determined.

In general, the third plane of projection is taken perpendicular to the ground line, the traces of the given planes upon it are determined, and thus the construction is referred to the general case. This solution is that given in the figure.

The traces of the third plane are *a a*, *a a′*, both ⊥ to *xy:* they meet the traces of the plane *ap′a* in *a* and *a′*; and, if we conceive the vertical plane of projection to be placed, for an instant, in its true position, the line joining *a* and *a′* will be the trace of the plane *apa′* upon the third



plane. Now make this third plane turn round aa till it coincide with the horizontal plane; the point a will not be changed, aa' will coincide with xy, and the point a will coincide with a'' if $aa''=aa'$, consequently, the line joining a and a' in space, will coincide with aa''.

In the same manner the trace bb'' of the other plane upon the third plane may be found; the point d, where it cuts aa'', is common to the two planes, and we have merely to determine the projections of this point upon the primitive planes.

Now, if dm be drawn $\perp$ to $a\,a$, its horizontal projection will be m: and if dd' be drawn $\perp$ to xy, and we take in aa', $am'=dd'$, its vertical projection will be m'. Join $\therefore$ pm and pm': these will be the projections of the intersection required.

4°. Let the planes be parallel to the ground line, then their traces and their intersection are also parallel to that line, and the general construction does not apply. We must in this case also use a third plane, as we have done above; but the mode of proceeding will be precisely the same as that above described, and will therefore be sufficiently intelligible to the learner without further description.

Problem 3.

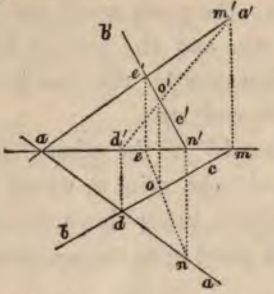
To find the intersection of a line with a plane.

17. Let a plane pass through the given line and cut the given plane: the point where the given line and the line of the planes' intersection meet will be the point required.

1°. Let the plane passing through the given line be vertical, that is $\perp$ to the horizontrl plane.

Let aa, $a'a$, be the traces of the given plane; $bcb'c'$ the projections of the line. Produce bc to m, draw $mm'\perp$ to

xy, meeting aa' in m'; then bm and mm' will be the horizontal and vertical traces of the plane through the given line; let bm meet aa in d: then d and m' are the intersection of the traces of the two planes; and if dd' be drawn ⊥ to xy, and $m'd'$ be joined, $m'd'$ will be the vertical projection of the intersection of the two planes. The vertical projection of the required point should be on $m'd'$; but it must also be upon the vertical projection $b'c'$ of the given line; it is ∴ at o', the point of their intersection.



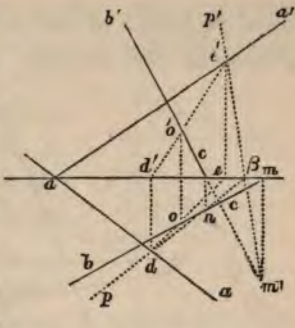
The horizontal projection o may be found by drawing $o'o$ ⊥ to xy, and meeting bc in o.

The point o may be found as we have determined o', by supposing the third plane to be ⊥ to the vertical plane. Its traces are $n'n$ and $n'b'$, and they cut those of the given plane in n and e; then draw $e'e \perp xy$, join en, it must meet bc in the point o.

2°. Let the plane passing through the given line have any position whatever.

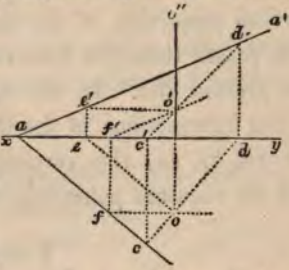
It is obvious that its traces will pass through those of the given line: first, therefore, determine the traces, n and m' of the given line, bc, $b'c'$; then through those traces, and any point β in xy, draw the lines βp and $\beta p'$. The given line is in the plane $p\beta p'$, since two points of it are in that plane.

Next, to determine the intersection of the two planes. From d and e', where the traces of the planes intersect, draw dd' and $e'e$ ⊥ to xy, draw the lines de and $d'e'$, these are the projections of the intersection of the two planes.



Now the projections of the required point ought to be in these projections and in those of the given line, bc, $b'c'$: hence the horizontal projection ought to be at o, the vertical projection at o', and ∴ the line oo' should be ⊥ to xy.

Cor. Suppose the given line to be ⊥ to one of the planes of projection,—for instance, the horizontal plane,—its horizontal projection will be a single point o, and its vertical projection $o'o''$ is ⊥ to xy.



The plane cdd' passing through it is vertical, and its trace dd' ⊥ to xy; but its horizontal trace has only one condition, namely, that of passing through the point o.

The other part of the construction is the same as in the general case, the vertical and horizontal projections of the required point being o' and o.

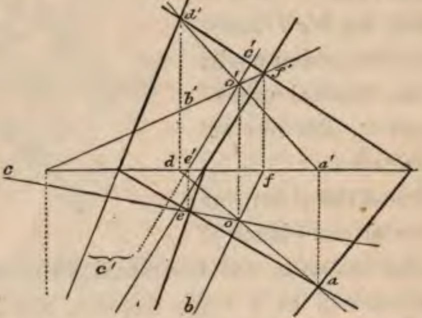
The preceding construction will enable us to solve the following problem:—Given one of the projections of a point situated in a given plane, to find the other projection.

Problem 4.

To find the points of intersection of three given planes.

18. The planes, taken two and two, give three lines, which pass through the required point. We therefore by Problem 2 find the projections of these lines; and, when the constructions are correctly made, the three horizontal projections intersect in the same point, which is the hori-

zontal projection of the required point; the three vertical projections also intersect in the same point, which is the



vertical projection of the same point; and, lastly, the line that passes throught hese two projections ought to be $\perp$ to the ground line. In the figure, the horizontal projections of the intersections of the three planes, taken two and two, are ad, bf, ce; the vertical projections are $a'd'$, $b'f'$, $c'e'$; and the required point has for its projection o and o'.

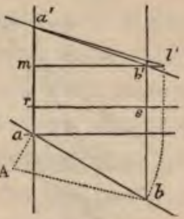
Problem 5.

Given the projections of two points, to find the projections of the line which passes through them, and the distance between them.

19. Let $a\,a'$ and $b\,b'$ be the projections of the points A and B. Draw the lines ab $a'b'$; these, it is evident, will be the projections of the straight line AB.

Next, to find its magnitude. We know that aa' and bb' are $\perp$ to xy, also that ra' and sb' are equal to the altitudes of A and B above the horizontal plane; if ∴ we conceive two vertical lines to be drawn, one from a and the other from b, and respectively equal to ra' and sb', their extremities will be the points A and B, and the distance between them is the distance required.

Since the two vertical lines are in the same plane with ab, conceive through B a line to be drawn parallel to ab, and terminated at the other vertical, a right-angled triangle will be formed, of which the base is $= ab$, and the altitude $=$ the difference between the verticals, and the hypothenuse is the distance required, AB.

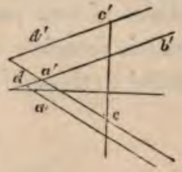


To construct this triangle, through b' draw $l'm$ parallel to xy, meeting ra' in m, take $ml' = ab$, draw $a'l'$; then $a'l'$ is the distance required.

Problem 6.

Given the projections of a straight line and a point, to find those of a line passing through the point, and parallel to the given line.

20. Let ab $a'b'$ be the projections of the line, cc' of the point; through c draw cd parallel to $a'b'$; through c' draw $c'd'$ parallel to $a'b'$: then cd and $c'd'$ are the projections required.



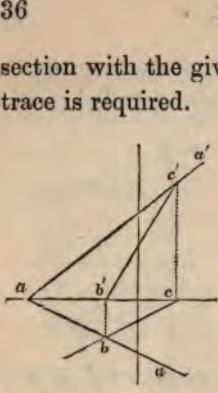
For when two lines are parallel, their projecting planes are also parallel, and ∴ the intersections of these planes with a third plane are also parallel; *i. e.*, the projections of two parallel lines are also parallel.

Problem 7.

Given the traces of a plane, and also one of the projections of a line in the plane, to find the other projection.

21. Suppose the horizontal projection of the line to be given; through it suppose a vertical plane to pass, its inter-

section with the given plane will be the line whose vertical trace is required.

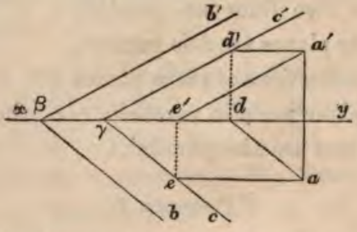


Let aaa' be the given plane, bc the given projection. Draw $cc' \perp$ to xy, it will be the vertical trace of the plane through cb; and b and c' where the traces of the two planes intersect, are points of the line which has bc for its horizontal projection. Draw $\therefore\ bb'$ $\perp$ to xy, join $b'c'$, it will be the vertical projection required.

Problem 8.

Through a given point, to draw a plane parallel to a given plane.

22. Let a, a' be the projections of the given point; βb $\beta b'$, the traces of the given plane. Then, since the intersections of two parallel planes with a third plane are also parallel, therefore the traces of the required plane must be parallel to those of the given plane.



Through the given point a, a', and in the required plane, suppose a line to be drawn parallel to the horizontal trace of the plane; it will $\therefore$ be parallel to βb; its horizontal projection ad must pass through a, and be parallel to βb;

its vertical projection must pass through a', and be parallel to xy.

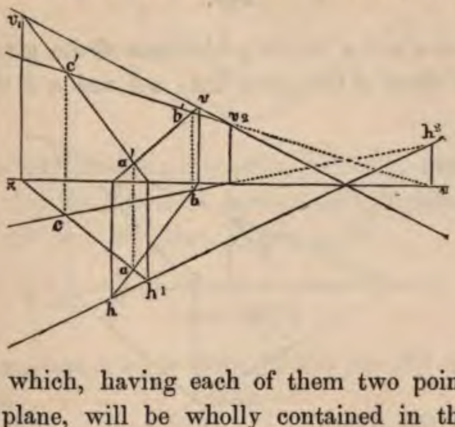
Let d' be the vertical trace of this line, it is ∴ a point in the vertical trace of the required plane; draw ∴ $d'c'$ parallel to $\beta b'$, then $d'c'$ is the vertical trace of the required plane: produce $c'd'$ to meet xy in γ, draw γc parallel to βb, it will be the horizontal trace of the required line.

To verify this construction, suppose through the given point a line to be drawn parallel to $\beta b'$, of which the projections are ae and $a'e'$; the point e, where it meets the horizontal plane, must be on the horizontal trace of the required plane.

Problem 9.

To draw a plane through three given points.

23. Since the points are given, their projections are also given; let these be a, a', b, b', c and c'. Join therefore the points, taken two and two, by the lines ab, $a'b'$, bc, $b'c'$,



ac, $a'c'$; which, having each of them two points in the required plane, will be wholly contained in that plane. Find, therefore, the three points of intersection of these lines with the planes of projection, and we shall have the two traces of the required plane; the three points of each trace must be in the same straight line, and the two traces must intersect the ground line in the same point.

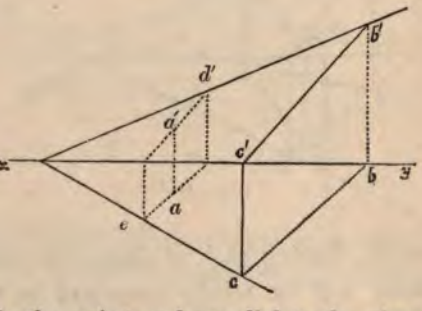
Thus let the line $a\,b$, $a'b'$ meet the planes of projection in h, and v; the line ac, $a'c'$, in h_1 and v_1; the line bc, $b'c'$, in h_2 and v_2; consequently the horizontal trace of the required plane will pass through the points h, h^1, h^2, and the vertical through v, v_1, v_2, and these two traces shall cut xy in the same point t.

Cor. If the line joining two of the points be parallel to one of the planes of projection, as, for instance, the vertical plane, the preceding construction could not be executed. But, in that case, the vertical trace of the required plane is parallel to that line; or, what is the same thing, to its vertical projection.

Problem 10.

To draw a plane through a given point and through a given line.

24. Let a and a' be the projections of the given point; bc and $b'c'$ those of the given line; ad and $a'd'$ those of a



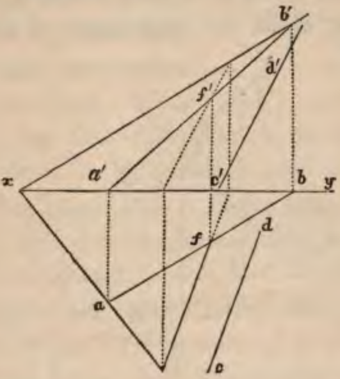
line through the point and parallel to the given line, and therefore situated in the required plane.

Let c and e be the horizontal, and b' and d' the vertical, traces of these lines; ∴ the lines ce, $b'd'$, are the traces of the required plane, and produced must meet xy in the same point.

PROBLEM 11.

Todraw a plane through a given line parallel to a given line.

25. In the first line ab, $a'b'$, take any point, f, f', and through it draw a line parallel to the second line cd, $c'd'$.



The required plane must contain this line, and the line ab, $a'b'$; and the traces of these lines will determine the traces of the required plane. (Prob. 9, 10.)

PROBLEM 12.

Through a given point to draw a plane parallel to two given lines.

26. Through the given point draw lines parallel to the given lines; the intersections of these lines with the planes of

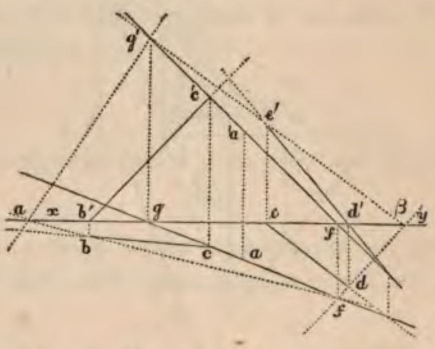
projection, will give two points of each trace of the required plane.

Problem 13.

To draw a line through a given point which shall meet two given lines.

27. First, find the traces of a plane passing through the point, and through one of the lines; then the traces of a second plane, also passing through the point and the other line; the line required, which ought to be in both planes at the same time, is therefore their common intersection.

Thus, let a, a' be the projections of the given point bc, $b'c'$, and de, $d'e'$, those of the lines; let fa, ag', be the traces of the first plane; $f\beta$, βg, those of the second plane: then gf, $g'f'$, will be the projections of their intersection, or of the line required.

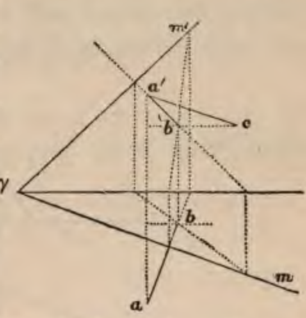


To verify this construction, we must have, 1°. That this intersection should pass through the given point. 2°. It should meet the first given line. 3°. It ought also to meet the second line.

Problem 14.

From a given point to draw a line perpendicular to a given plane, and then to determine its magnitude

28. We shall first show that the projections of the perpendicular are ⊥ to the traces of the given plane. The plane which projects the line on the horizontal plane is ⊥ both to the horizontal and the given plane (Prop. XIX., Introduction); ∴ conversely, both are ⊥ to the projecting plane, and consequently their intersection is also ⊥ to it (Prop. XX., Introduction); but this intersection is the horizontal trace of the given plane, ∴ this trace is ⊥ to all lines situated in the projecting plane, and ∴ to the horizontal projection of the perpendicular. A similar method of reasoning applies to the vertical projection.



This being the case, draw, through the projections a, a' of the given point, ab, $a'b'$, respectively, ⊥ to the traces γm, $\gamma m'$ of the given plane; and it is manifest we shall have the projections of the required perpendicular. We can then determine b and b', (Prob. 3,) the projections of the foot of the perpendicular, and, lastly, we can find its length $a'c$, (Prob. 5.)

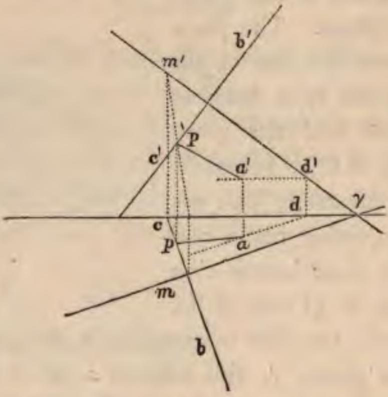
Problem 15.

Through a given point to draw a line and plane, both perpendicular to a given line.

29. First find the traces of a plane ⊥ to the given line; these are ⊥ to the projections of this line, consequently it

will be sufficient to know one point of either of these traces to know both.

Let a, a' be the projections of the given point, bc, $b'c'$ those of the given line. Through a, a', suppose a horizontal line to be drawn, parallel to the horizontal trace of the required plane; its horizontal projection being parallel to this trace will be ⊥ the horizontal projection bc of the given line; and thus this projection is a known line ad: the line $a'd$, parallel to xy, will be the vertical projection of this line. And as this horizontal line is altogether in the required plane, therefore d', its intersection with the vertical plane, will be a point of the vertical trace of the required plane. Draw, therefore, through this point $\gamma m'$ ⊥ to $b'c'$: then γm ⊥ to bc, and $m \gamma m'$ is the plane required.



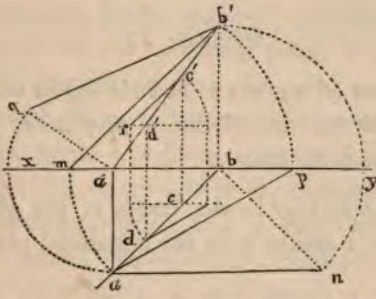
Next, to find the projections of the ⊥ from the given point upon the given plane. To do so, find p and p', the projections of the intersection of this line with the plane $m\gamma m'$: draw the lines ap and $a'p'$: these are the projections required.

PROBLEM 16.

Given the projections of a line, to find the angles which it makes with the planes of projection.

30. DEF. The angle which a line makes with a plane is the angle formed by the line and its projection upon the plane.

Let ab and $a'b'$ be the projections of a line, aa' and bb' its traces, determined by a previous proposition; let the plane abb' turn round bb', till it coincide with the vertical plane; during this revolution the line ab is not altered in magnitude, and as it is always $\perp bb'$, it will take a position as bm along xy. At the same time the line joining a and b' in space will coincide with $b'm$, and thus the required angle coincides with the angle $b'mb$.



If the plane $aa'b'$ be supposed to revolve round ab', the angle might be found in the horizontal plane; in this case the vertical bb' would be measured along bn, $\perp$ to ab, and the required angle would be nab.

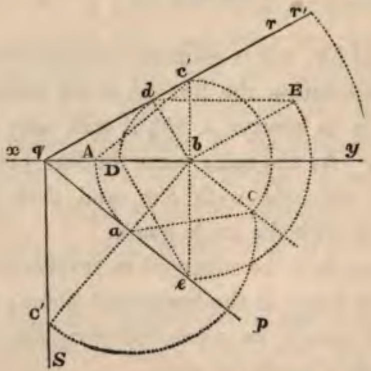
To find the angle which the line makes with the vertical plane, a similar construction must be made; it is represented by apa' on the horizontal, or by $qb'a'$ on the vertical plane.

When the two points a and b' are very distant, we may use the following method:—Take any two points in the line; let c, c', d and d', be their projections. Through the point dd' imagine a line drawn parallel to ab, and find the angle it makes with the line. To do this, let the vertical plane which contains the angle turn round the vertical from c, till the plane is parallel to the vertical plane of projection; then the required angle is projected on this plane in its true magnitude, and is represented by $c'rd$. We proceed in the same manner to find the angle with the vertical plane.

Problem 17.

Given the traces of a plane to find the angles which the plane makes with the two planes of projection, and also the angle between the traces.

31. Let qp and qr be the traces of the plane; to find first the angle it makes with the horizontal plane.



Since the angle between two planes is measured by that between two perpendiculars, drawn, in each of the planes, from the same point of their common intersection; draw

$ab \perp$ to pq, and $bc \perp$ to xy, meeting qr in c; the straight line joining c and a is in the given plane, and is $\perp pq$; and the $\angle$ it makes with ab is the $\angle$ required. (Prop. XI. Introduction.) The $\angle$ is that of a right-angled $\triangle$, of which ab is the base, and bc the height. To make it coincide with the vertical plane, suppose it to revolve round bc in xy; take $b\mathrm{A} = ba$, and $\angle c\mathrm{A}b$ is the angle required; and for the horizontal plane, draw $b\mathrm{c} = bc'$ and $\perp$ to ba, and $\mathrm{c}ab$ will be the angle required.

To find the $\angle$ the plane makes with the vertical plane draw $bd \perp qr$, and $be \perp$ to xy. The line in space, joining d and e, makes with db the required $\angle$; and if this $\angle$ be made to coincide with the horizontal plane, it is represented by $e\mathrm{D}b$; if with the vertical plane, by $\mathrm{E}db$.

Lastly, to find the angle between the traces. Since the line joining the points a and c' is $\perp$ to pq, if the plane pqr turn round pq till it coincide with the horizontal plane, ac' will coincide with $a\mathrm{c}'$, which is $\perp$ to pq; the distance aq remaining the same; the point $c'\mathrm{c}'$ comes to c', $a\mathrm{c}'$ being made equal to $a\mathrm{c}$ or to $c'\mathrm{A}$, the real length of the line joining the points $a\gamma c^{1}$; qr will therefore coincide with qs, drawn through q and c', and pqs is the $\angle$ of the traces.

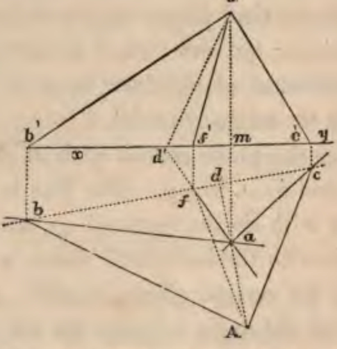
PROBLEM 18.

Given the projections of two lines, to find the angle between the lines.

32. We may remark that, when the points of intersection of the two horizontal projections of the lines, and of their two vertical projections, are not in the same perpendicular to the ground line, the two lines do not really meet. In that case we draw from any point two lines parallel

to them, and the angle between these lines is the angle required

Let $\therefore$ a and a', ab and $a'b'$, ac and $a'c'$, be the projections of this point and of the two parallel lines.

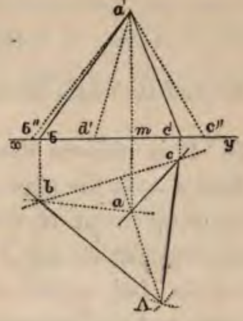


Find b and c, the intersections of the two lines with the horizontal plane, and join $b\,c$; we may consider this line as the base of a $\triangle$, of which a and a' are the projections of the vertex. Our object is to find the vertical $\angle$, and we must $\therefore$ construct the $\triangle$.

Draw $\therefore$ $a\,d \perp$ to $b\,c$, and suppose a line drawn from d to the vertex $a\,a'$; this line will be $\perp$ to $b\,c$; so that if the $\triangle$ revolve round $b\,c$, till it coincides with the horizontal plane, this $\perp$ will be in the direction of $d\,a$; and to find its length, we see that it is the hypothenuse of a right-angled $\triangle$, of which the sides are $= a\,d$ and $a'm$; $\therefore$ on xy, take $m\,d' = a\,d$, and $a'd'$ will be this hypothenuse.

Then produce $d\,a$ to A, so that $d\text{A} = a'd'$; join $b\,\text{A}$, $c\text{A}$, and $\angle\ b\,\text{A}\,c$ will be the $\angle$ required.

We have in the construction of the $\triangle$ used its altitude; we might have employed its sides. For the horizontal projections of the sides are $a\,b$ and $a\,c$, their vertical are $a'b'$ and $a'c'$; then make $m\,b'' = a\,b$, $m\,c'' = a\,c$; draw the hypothenuse $a'b''$, $a'c''$; these are the lengths of the lines. With centres b and c, and with radii equal to $a'b''$, $a'c''$, describe two arcs; their intersection determines the vertex A, and consequently the $\angle\ b\,\text{A}\,c$.



Problem 19.

To bisect the angle between two given lines.

33. Let $a\,b$, $a'b'$, $a\,c$, $a'c'$, be the projections of the two lines. (See last fig. but one.)

1°. As in the preceding proposition, construct the $\angle\ b\,\text{A}\,c$ between the lines, and draw Af bisecting this angle. It is manifest that, if the $\triangle\ b\,\text{A}\,c$ be raised to its first position, the line Af will still bisect the angle: we must ∴ find the projections of Af.

Now, during the motion of the $\triangle$, the point f, where the line cuts $b\,c$, does not alter its position; it is ∴ its horizontal projection; and if ff' be drawn $\perp$ to xy, f' will be its vertical projection. One of the projections ∴ of the required line should pass through f, and the other through f': but they ought also to pass through a and a'; ∴ af and $a'f'$ are the projections of the line.

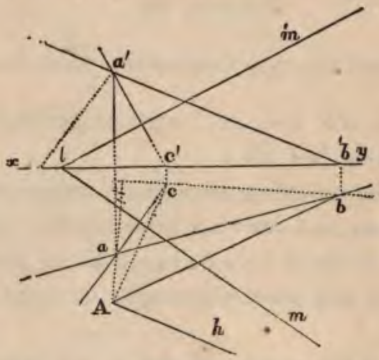
If, instead of bisecting the angle, it be required to divide it into two parts, having a given ratio to each other, we must draw Af so that $\angle\ b\,\text{A}\,c$ may be similarly divided. The rest of the construction remains the same.

Problem 20.

To find the angle between a straight line and a plane.

34. It is obvious that, if from any point in the line we draw a $\perp$ to the plane, the $\angle$ between these two lines is the complement of the required angle. We have $\therefore$ merely to construct the $\angle$ between these two lines.

Let lm, lm', and ab, $a'b'$, be respectively the traces of the plane and the projections of the line.



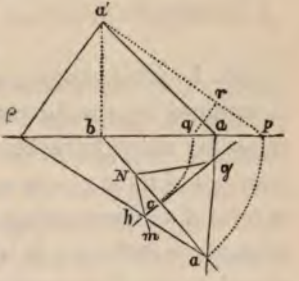
From a draw aa' $\perp$ to xy; draw ac, $a'c'$ $\perp$ to lm and lm'; then ac and $a'c'$ will be the projections of a $\perp$ from a point in the line on the plane. Then construct the $\angle$ between this $\perp$ and the given line; and the $\angle$ $b\text{A}c$ thus found will be the complement of the angle required; then draw $\text{A}h$ $\perp$ to $\text{A}c$, and the required $\angle$ will be $b\text{A}h$.

Problem 21.

To construct the angle between two planes.

35. Let $a\alpha a'$, $a\beta a'$ be the two planes; find ab, the horizontal projection of their intersection: suppose them to

be cut by a plane $\perp$ to their intersection, the traces of this plane are $\therefore$ $\perp$ to the projections of that line (Prob. 14); draw $\therefore$ $gch \perp ab$, gch may then be taken for the horizontal trace of the plane $\perp$ to the intersection.



This plane cuts the given planes in two lines, which pass, the one through g and the other through h, and which make with gh a $\triangle$, in which the $\angle$ opposite to gh is the $\angle$ required. For the $\angle$ between two planes is measured by the $\angle$ between two lines drawn from any the same point of the common intersection of the two planes, and $\perp$ to it, one in one plane, the other in the other.

The vertex of this $\triangle$ is in the vertical plane aba'; now this plane is $\perp gh$ $\because$ gh is $\perp ab$; $\therefore$ the line from c to the vertex of the $\triangle$ is $\perp gh$; $\therefore$ by being made to coincide with the horizontal plane, it lies in the direction of cb.

To find its length: since the plane of the $\triangle$ is $\perp$ to the intersection of the given planes, and the line required is in the plane of the $\triangle$, it is $\therefore$ $\perp$ to this intersection. Let the vertical plane aba', revolve round its vertical trace ba', until ba coincides with xy.

The points a and c will describe the arcs ap and cq, round b, and will come to p and q. Then the intersection of the given planes will be in $a'p$, and the required distance will be the line qr, $\perp$ to $a'p$: make cN in ab, therefore, $= qr$, draw the lines Ng, Nh, and $\angle h$Ng is the $\angle$ required.

CURVE SURFACES AND TANGENT PLANES.

36. In what has preceded, the required points have been determined by the intersection of planes, and the questions regarding them have been considered as solved when the positions of these planes are determined. We now proceed to treat of a different class of problems, which are dependent upon curve surfaces; it is therefore necessary to make some remarks upon these surfaces.

We know that when a curve is traced upon a plane, it may be considered as a series of points connected together and determined by a law, which furnishes what the geometers call the equation to the curve: thus, the circle furnishes us with an instance of this law, since every point in it is at the same distance from one point, called the centre of the circle. This property gives us both a mechanical method of describing the circle, and a geometrical relation by which its equation may be found. Now, surfaces differ from curves in this respect: that their respective points are in space, and are not limited in position to the same and constant plane, but extend in all directions. Still these points are subject to a law, which particularizes and determines the surface in question.

The curves just spoken of being always in the same plane are called plane curves. There are curves called curves of double curvature, which are not in the same plane: such is the helix, of which the corkscrew is a familiar instance. We might trace such a curve by the motion of a point, and thus say generally, that curves are described by the motion of a point, while surfaces are described by the motion of a line.

37. The surfaces of which we shall treat may be divided into—1, Surfaces of Revolution; 2, Conical Surfaces; 3, Cylindrical Surfaces.

Surfaces of Revolution are those which may be supposed to be described by the revolution of a curve round its axis. Thus the semicircle revolving round its diameter produces the sphere, and the ellipse round its axes generates the spheroid. From this method of generation, it is plain that every section perpendicular to the axis is a circle; while every section made by a plane passing through the axis will reproduce the original curve.

Conical surfaces are described by the motion of a line, which is compelled always to pass through a fixed point, and the other extremity moves through a curve traced upon a given plane. If the given curve be a circle, and if the given point be in a line passing through the centre of the circle, and perpendicular to its plane, the surface described is that of the common, or right, cone.

Cylindrical surfaces are described by the motion of a line always parallel to its first direction, but the extremity moving through a given curve in a given plane.

If the given curve be a circle, and the line be perpendicular to the plane of the circle, the surface described is the common, or right, cylinder.

DEF. The moveable line which generates the surface is called the generatrix, and the line which directs and limits its motion the directrix.

THE TANGENT PLANE AND THE NORMAL.

38. We may consider a curve to be a polygon, of which the sides are infinitely small, and one of its tangents to be the prolongation of one of these indefinitely small sides. In the same manner a curve surface may be considered as a polyhedron, whose faces are infinite in number and infinitely small, and the tangent plane is the plane of one of these faces. And to have a clear idea of a tangent plane to a point in a surface, we must suppose that round this point there has been taken an infinitely small portion of the surface, which is considered a plane, and this plane, indefinitely extended, is the tangent plane.

39. If we conceive through the point of contact any number of lines whatever to be traced upon the surface, and if from the point under consideration we take infinitely small portions of these curve lines, we may regard these portions as straight lines, which produced will be tangents to the respective curves of which ultimately they form a part; but all these small lines will also be in the tangent plane, and therefore when produced will also be in that plane.

Hence we see that the tangent plane to a surface contains all the tangents, drawn through the point of contact, to all the lines which can be traced through the point upon the surface.

40. This proposition is of great importance in determining the position of a tangent plane, since as it is only necessary to have two lines given in order to find a plane, it follows, that in order to construct the tangent plane to a

given point of a surface, we have merely to draw tangents to *two lines traced upon the surface*, and to make the plane pass through these tangents. Our choice of the lines on the surface will, of course, be directed to those whose tangents are most readily found.

Def. A normal is a perpendicular to the tangent plane, drawn through the point of contact.

TANGENT PLANES TO SURFACES.

THE CYLINDER.

PROBLEM 1.

Given the horizontal trace of a cylinder and the direction of the generatrixes, to draw a tangent plane to a given point of the cylinder.

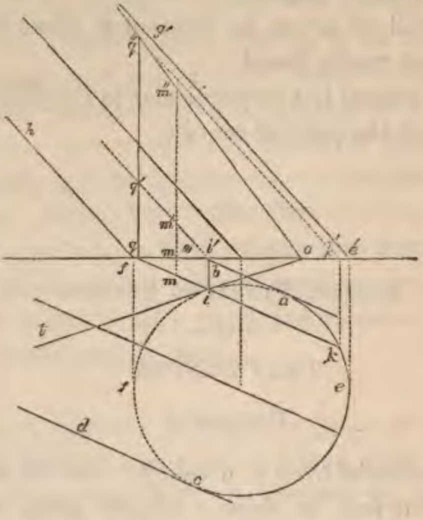
41. Let *aecf* be the horizontal trace of the cylinder; and in this instance suppose *aecf* to be a circle.

Draw *ab*, *cd*, tangents to *aecf*, and parallel to the horizontal projections of the generatrixes; these tangents will be also the limits of the horizontal projection of the cylinder, since all the generatrixes must be projected between these two lines.

Draw the tangents ee', ff', ⊥ to xy, and $e'g'$, $f'h'$, parallel to the vertical projection of the generatrixes; it is clear that these latter lines will be the limits of the vertical projection of the cylinder.

Now, suppose that the horizontal projection *m*, of a point in the cylinder be given, and let the other projection be required.

Draw through m a vertical plane, parallel to the generatrixes, which can intersect the cylinder only in one or more of



the generatrixes; the intersections of these generatrixes with the vertical, drawn through the point m, give the points of the surface which are projected in m. The traces of this plane are the line $m\,q$, drawn parallel to $a\,b$, through the point m, and the line $qq' \perp$ to xy. And since the first meets the curve $aecf$ in i and k, it follows that the plane cuts the cylinder in two generatrixes, the horizontal projection of both being in the direction of mq; and their vertical projections are found by drawing ii', kk', $\perp$ to xy, and $i'q'$ $k'q''$ parallel to $f'h'$.

Draw $mm' \perp$ to xy; and m', m'', where mm' meets $i'q'$, and $k'q''$, are the vertical projections of points of the cylinder corresponding to the horizontal projection m.

This being done, next to draw a tangent plane to the point $m\,m'$. We see, first, that this plane ought to contain the generatrix, $i\,m$, $i'm'$, whose traces are i and q'; and farther, it must be a tangent to the cylinder at every point of this generatrix. Now the tangent plane to a surface contains the tangents to every curve traced upon this surface, and passing through the point of contact; if ∴ we draw the tangent ti, to the curve aec, it will be in the required plane, and since it is in the horizontal plane, it is the horizontal trace of the tangent plane. Produce $i\,t$ to meet xy in o; draw oq''; $o\,q''$ will be the vertical trace of the plane.

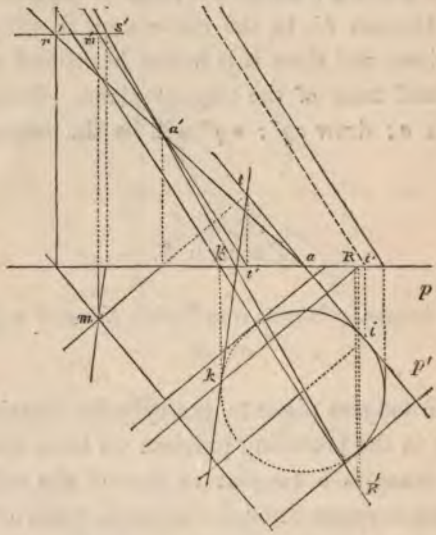
PROBLEM 2.

To draw a tangent plane to a cylinder, through a point without it.

42. The tangent plane to the cylinder contains a generatrix, and in the preceding problem we have seen that its horizontal trace is a tangent to that of the cylinder: if, therefore, we suppose through the given point a line to be drawn parallel to the generatrixes, it will be altogether in the tangent plane; and if, after having constructed the horizontal and vertical traces of this parallel, we draw through the former, tangents to the horizontal trace of the cylinder, these tangents will be the horizontal traces of tangent planes, passing through the given point. Afterwards, to find their vertical traces, join the vertical trace of the same parallel to the points where the ground line is intersected by the horizontal traces of the planes. The details of the construction are represented in the next figure.

The projections of the cylinder are traced, as in the preceding proposition; the projections of the given point are m and m'; those of the parallel to the generatrix are

mt and $m't'$; the trace of the parallel is t and t': therefore the tangents ta, $t\beta$, drawn to the horizontal trace of the cylinder, will be the horizontal traces of the tangent planes, and aa', $\beta a'$, will be its vertical traces.



To verify the constructions, we have drawn through the given point lines parallel to ta and $t\beta$: the points r' and s', where the parallels meet the vertical plane, should be on aa' and $\beta a'$. The generatrixes in which the planes touch the cylinder, serve the same purpose. One of them, which has its projections ip and $i'p'$, meets the vertical plane in a point p': and the other, which has projections kR, and k'R$'$, meets it in R$'$. Now it is evident that the trace aa' must pass through the point p', and the trace $\beta a'$ through the point R$'$.

Problem 3.

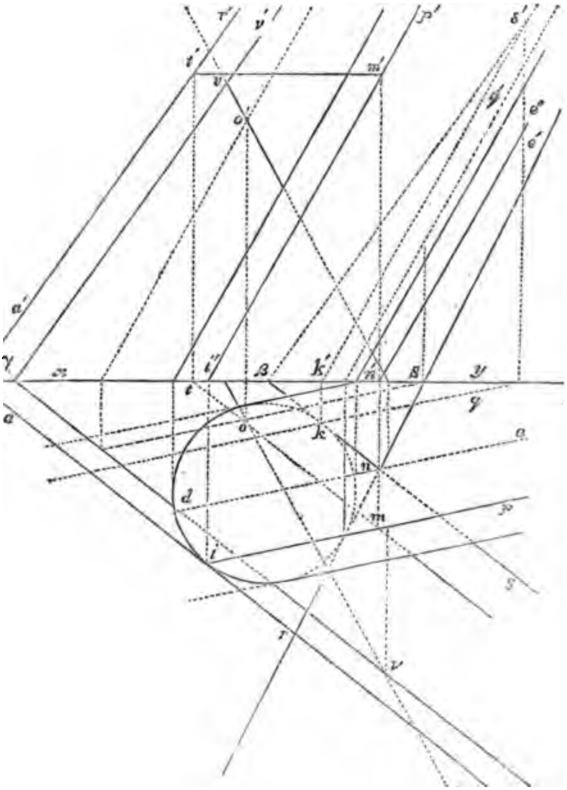
To draw a tangent plane to a cylinder, which shall be parallel to a given line.

43. A tangent plane, parallel to a given line, ought to contain a generatrix of the cylinder, and a parallel to the given line. If, therefore, we draw through a point of this line a parallel to the generatrixes, the tangent plane will be parallel to the plane determined by these two lines. Consequently, we construct the traces of this latter plane, and draw, parallel to its horizontal trace, tangents to the horizontal trace of the cylinder; these tangents will be the horizontal traces of the tangent planes, which satisfy the conditions of the problem. The vertical traces may be constructed, since we know the points where they cut the ground line, and a line to which they are parallel.

In the fig., the point $o\,o'$, taken upon the given line, is that through which the parallel to the generatrixes of the cylinder is drawn; and the traces of the plane passing through these two lines are $\gamma\,\nu$ and $\gamma\nu'$: the tangents $r\,a$ and $s\beta$ are drawn parallel to the horizontal trace $\gamma\nu$, and ar' and $\beta s'$, parallel to $\gamma\nu'$: the planes rar', $s\beta s'$, are the tangent planes required.

To verify the constructions, we observe that the generatrixes ip, $i'p'$, and $k\,q$, $k'q'$, ought to meet the vertical plane in the traces ar' and $\beta s'$.

When the point $p\,p'$, is very distant, it would be inconvenient to determine $k'q'$. In this case take in the generatrix im, $i'm'$, the point m,m', and through it conceive a line parallel to ra: its projections are mt, parallel to ra, and $m't'$ parallel to xy: and since it is in the tangent plane, its vertical trace *t′* is on the vertical trace of the plane. Hence the point *t′* can serve to determine the latter trace, or to verify it when it is known. With the same view we could draw a line parallel to the generatrixes, through any point in the horizontal trace ***ra***.



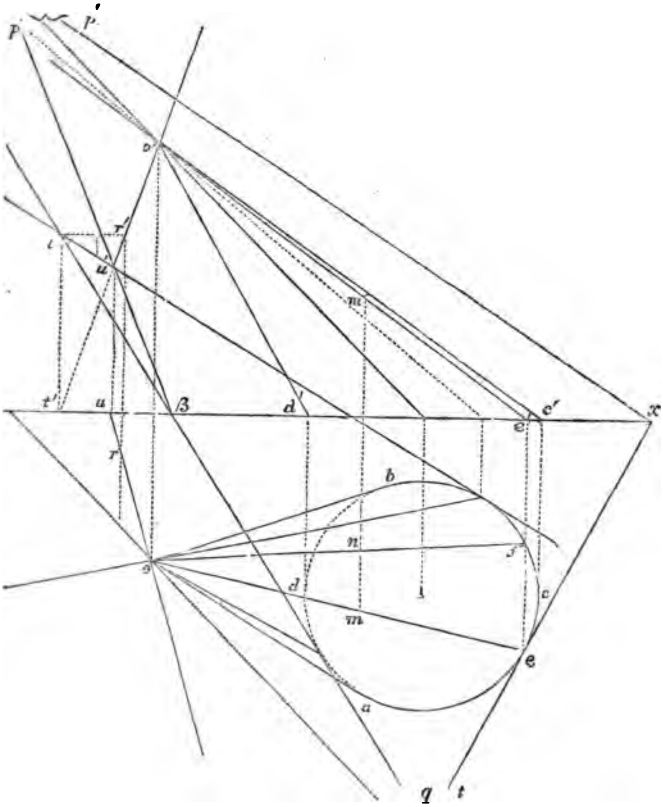
If we draw a second tangent plane to the cylinder, its intersection with the former ought to be parallel to the generatrixes, since each tangent plane contains a generatrix, and the generatrixes are parallel each to each.

THE CONE.

Problem 4.

Given the horizontal trace of a cone and the projections of its vertex, to find the tangent plane to a point in the surface.

44. Let *acd* be the base or horizontal trace of the cone, *o,o′* the projections of the vertex.



To find the outline of the cone, or the limits between which all the generatrixes are projected, draw in the horizontal plane the tangents oa and ob; then the tangents cc', dd', ⊥ to xy: and the lines $o'c'$, $o'd'$, in the vertical plane.

First, to determine the points of the surface which correspond with a given vertical projection, m'. Draw the line $o'm'$, cutting xy in e', and $e'e$ ⊥ to xy; then $o'e'$, $e'e$, are the traces of a plane ⊥ to the vertical plane, which contains the required points, and which passes through the vertex of the cone. This plane can only meet the cone in its generatrixes; and as its horizontal trace ee', meets the base acd, in points e and f, it is evident that oe and of are the horizontal projections of these generatrixes; thence it is easy to find the horizontal projections m and n of the required points.

Suppose that the tangent plane to the point mm', be required. We observe that the plane ought to contain the generatrix oe, $o'e'$, and touch the cone in the whole extent of that line. Next, draw the tangent ex, to the point e, join xp', p' being the point where the generatrix meets the vertical plane, then ex and xp' will be the traces of the tangent plane.

Problem 5.

To draw a tangent plane to a cone, through a point without the cone.

45. Every tangent plane to a cone passes through its vertex, and its horizontal trace is a tangent to its base.

Draw, therefore, a line joining the given point and the vertex of the cone; and from the point where this line meets the horizontal plane, draw tangents to the base of the cone. These tangents will be the horizontal traces of the tangent planes required. To find their vertical traces, join the vertical trace of the same line to the points where the tangents to the base meet the ground line.

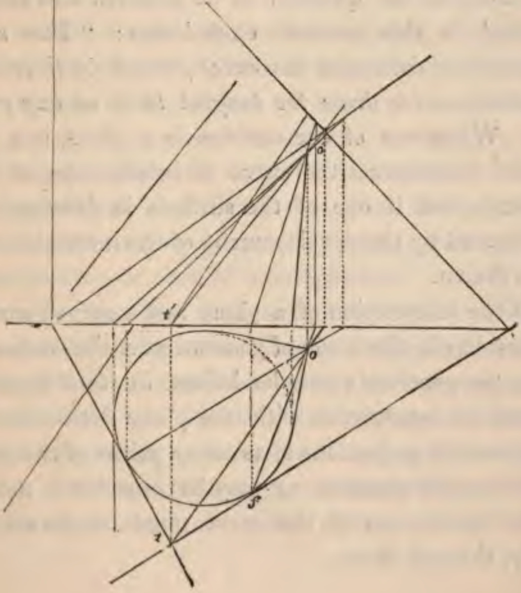
In the figure (see last fig.) the projections of the vertex are *oo′*; those of the given point are *rr′*; those of the line passing through these two points are *or* and *o′r′*; the traces of this line *t* and *u′*, and the tangent planes, are $u'\beta$ and $t\beta$.

To verify the constructions, we may find the vertical traces of the generatrixes of contact, and also the vertical traces of the parallels, drawn through the given point, to the horizontal traces of the two tangent planes.

PROBLEM 6.

To draw a tangent plane to a cone, parallel to a given line.

46. The tangent plane, which passes through the vertex, must contain a line drawn from the vertex parallel to the given line. This will be sufficient to shew that, after having traced the projections *ot*, *o′t′*, of this parallel, the constructions will be the same as in the preceding problem.



CURVES AND THEIR TANGENTS.

General Principles.

A line is defined and determined, either by some property which characterises it, or as being produced by the motion of a point, or as the intersection of two surfaces, each of which contains it. In the greater number of cases, in the practical application of descriptive geometry, lines have to be considered as being produced by the intersection of surfaces; and it is chiefly under this point of view that they will be treated of in this part. Nevertheless, the helix and the epicycloid will be investigated as examples of curves of the two former kinds. In the other examples the question to be resolved will always be comprised in this general enunciation: "*Two surfaces being given, to determine the curve formed by their mutual intersection and to draw the tangent to it at any proposed point.* When one of the surfaces is a plane, it is further necessary to construct the curve of intersection, of its true magnitude, and if one of the surfaces is developable, the curve formed by the developement of the intersection must also be drawn.

Let the intersection of a plane and a curved surface be first considered: the mode of generation of the surface being known, the generatrix may be defined in its different positions and its intersection with the plane determined. By this means the projections of as many points of the required section may be obtained as may be considered necessary, and the projections of the curve must be drawn, continuously, through them.

When it is the intersection of two curved surfaces that is required, the following is the general mode of proceeding. The surfaces are considered to be cut by a series of parallel planes, each of which determines by its intersection with the given surfaces two curves, the projections of which must be constructed, the points common to these curves obviously belong to the curve sought. It might at first be supposed that four curves would have to be drawn for each cutting plane, two on each plane of projection: but since the cutting planes, may be assumed in any position, they may always be taken as parallel to one plane of projection, consequently the projections of the sections made by these planes with the surfaces on the other plane of projection will be straight lines parallel to the ground line, or will be the *trace* of each cutting plane. The points on this straight line common to the intersection are immediately obtained from the projections of the curves on the other plane of projection.

In certain cases of frequent occurrence, the choice of these auxiliary sectional planes is determined by the nature of the surfaces: thus, if one of them be a cylinder, the planes should be assumed as parallel to the axis of the solid, because in that case the intersections with the surface will be right lines, and if the two given surfaces are both cylindrical the cutting planes may be assumed parallel to the generatrixes, or axes, of both cylinders.

If one surface be a cone, and the other a cylinder, the planes should be assumed as passing through the vertex of the former, and as parallel to the axis of the latter solid.

If one of the given surfaces be a surface of revolution, the auxiliary sectional planes should be assumed as perpendicular to the axis of revolution, and the sections with this solid will, consequently, be circles (37).

It must be borne in mind that the tangent-plane at any point of a curved surface, contains all the tangents which can be drawn at that point to any curve lying in the surface, passing through the point. Hence it follows, that to draw a tangent at any point of a curve, resulting from the intersection of a given plane with a curved surface, the tangent-plane to the surface at that point must be constructed, and the intersection of the first-mentioned plane with this tangent-plane, will be the tangent to the curve. And, generally, if a line be the intersection of any two surfaces the intersection of the two tangent planes drawn, one to each surface at any point of the intersection of the surfaces, will be the tangent to that intersection at the point in question.

The tangent to a curve may be also determined from other principles. The normals (Def., p. 53,) drawn to two surfaces from any point of their common intersection, are respectively perpendicular to the two tangent-planes at that point: hence the required tangent to that intersection at that point, will be perpendicular to the plane passing through the normals.

This mode of proceeding is preferable to the former when the surfaces are surfaces of revolution, for in this case the normals are more readily drawn than the tangent-planes.

The projection of a tangent to a curve, is always a tangent to the projection of that curve; for if the tangent be considered as the prolongation of an element of the curve, the projection of that tangent will be the prolongation of the projection of that element.

It must be observed that in those cases, when from the particular conditions of the case the foregoing principles cannot be applied, the required tangent can only be drawn mechanically to touch the projection of the curve at the given point.

§ 1. SECTIONS OF CURVED SURFACES BY PLANES.

PROBLEM 1.

Intersection of a cylinder perpendicular to the horizontal plane by a plane perpendicular to the vertical plane.

47. The construction will consist of three distinct parts.

First, To determine the projections of the intersection and those of its tangents.

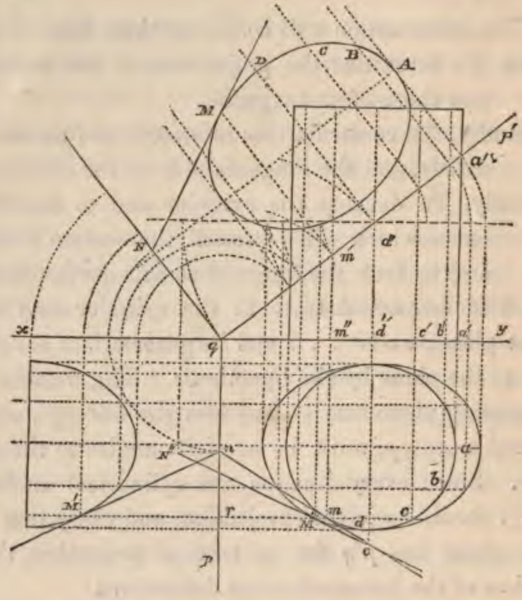
Secondly, To construct this intersection of its true magnitude, and the tangents to it at the same points.

Thirdly, To develop the cylinder and to describe the curve which will represent the section developed, and to draw the tangents to this developed curve.

1°. The horizontal trace of the cylinder may be any curve at pleasure abc . . . and its generatrixes are perpendicular to the plane by the hypothesis. The vertical trace of the cutting plane may be any straight line qp', while its horizontal trace qp, must be at right angles to the ground line xy. Since every line on the cylindrical surface has abc . . . for its horizontal projection, and every line on the cutting plane has $p'q$ for its vertical projection, the two *projections* of the intersection are determined.

Take any point m in abc . . . at pleasure, and through it draw $mm''m'$ perpendicular to xy to cut $p'q$ in m'; it is obvious that $m''m'$ is the vertical projection of the generatrix which passes through m, and that the point in which this generatrix cuts the plane pqp' has m, m' for its projections. m,m' are, therefore, the projections of a point in the curve of intersection. By repeating this construction, the projections of as many points may be obtained as are requisite.

The tangent at any one of these points, as for example m,m', is easily obtained. The tangent plane to the cylinder at this point is vertical, and has for its horizontal trace the tangent mn, to the base $abc \ldots$. Now the tangent required is the intersection of this tangent-plane with the given one; the projections therefore of this tangent are mn and qp'.

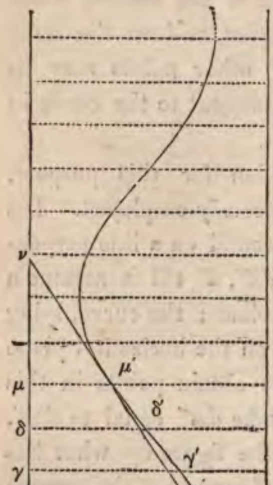


2°. To determine the true magnitude and form of the section let us suppose the plane pqp turned down on one of the planes of projection by rotating on the trace qp'. The point m, m' is situated in space on a perpendicular to qp' at a distance from m' equal to mm'': hence by drawing mM at right $\angle^s$ to qp' and making it equal to mm'', M will be one point of the curve sought. By the same construction as many points may be found as may be deemed necessary to allow the curve ABC . . . being drawn. To draw the tangent to this curve, at the point M, it is only necessary to observe, that in the rotation of the plane the line qn will fall in qN, perpendicularly to qp', so that by drawing MN, we obtain the tangent at the point M.

If the plane pqp' be turned round on its horizontal trace pq, the curve of the section may also be drawn of its true magnitude in the horizontal plane of projection. Draw mr perpendicular to pq; if we then conceive a line in space between r and the point m, m', this line will be perpendicular to pq and will be projected of its true magnitude in qm': if then $M'r$, in mr, produced, be made equal to qm', the point M' will be one in the curve of the section, when turned down on the plane of projection, and on the same principles any requisite number of other points may be obtained; nM will obviously be the tangent to the curve in mm' when this curve is turned down.

There is yet another construction for this purpose, which may be occasionally advantageously employed. Let the plane pqp be supposed to turn round on a line perpendicular to the vertical plane, as on dd'', d' till it assume a position parallel to the horizontal plane; the curve, lying in this plane, will then be projected on the horizontal plane of its true form and magnitude: to obtain points in this projection, it is only necessary to make dM'' equal to $d'm'$, as will be readily understood from the figure by what has been already explained. The point of contact of the tangent is now in M''; to draw the tangent at that point it is only necessary to deduce a second point in it, as N'', on the same principle as that by which M'' is obtained, the line nN'' being drawn parallel to xy for the horizontal trace of the plane, in which the trace of the tangent lies during the rotation of the plane pqp' on dd'', d'. N'' will consequently lie in nN'' thus drawn. It will also be observed that the point o in which the tangent nm, qm' cuts the axis of rotation dd'', d', not being changed during the rotation of the plane, the tangent $N''M''$ will pass through o.

3°. If a cylinder be considered as a prism with an infinite number of faces, it is clear that if this surface be developed, or *unrolled*, every section perpendicular to the generatrixes will become a right line on the developement at right angles to the edges, or arrises, of the prism when developed; and the length of these arrises, comprised between the oblique and perpendicular sections will not be altered. If therefore the curve abc . . . be divided into any number of arcs ab, bc, ed, &c., and a line be drawn, and made equal to the curve abc by setting off along it the equi-distances $a\beta$, $\beta\gamma$, $\gamma\delta$ . . . respectively equal in number and magnitude to the arcs ab, bc, cd, . . . then by setting the true lengths of the respective arrises along perpendiculars, aa', $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ . . . to the line a . . . δ, the curve $a'\beta'\gamma'$ drawn through these points will be the developement of the curve of the oblique section. The true lengths of the arrises are furnished by their projections $a'a''$, $b'b''$. . . . on the vertical plane.



Let the point m, m' be represented by μ' on the developed curve; then, since the elements of the original curve do not change their angle of inclination to the generatrixes, or arrises, by being developed, the tangents, which are these elements produced, preserve their inclination to the arrises when developed. Now the tangent at the point mm' passes through n, and the angle made by this tangent with the arris of the cylinder is an angle of the right-angled triangle, of which mn is the base, and $m''m'$ the altitude: if, therefore, $\mu\nu$ be made equal to mn, and $\mu'\nu$ be joined, this line will be the tangent to the curve.

Scholium. In the figure the curve $abc\ldots$ is a circle, but the construction above described is not the less general: whatever may be the curve $abc\ldots$, it will always be found that the tangents in m and M', to the curves $abc\ldots$ $\text{ABC}\ldots$, cut the trace pq in the same point n, or, which is the same, the tangents in m and M' to the curves $abc\ldots$ $\text{ABC}\ldots$, cut the right line dd'' in the same point o; hence this remarkable

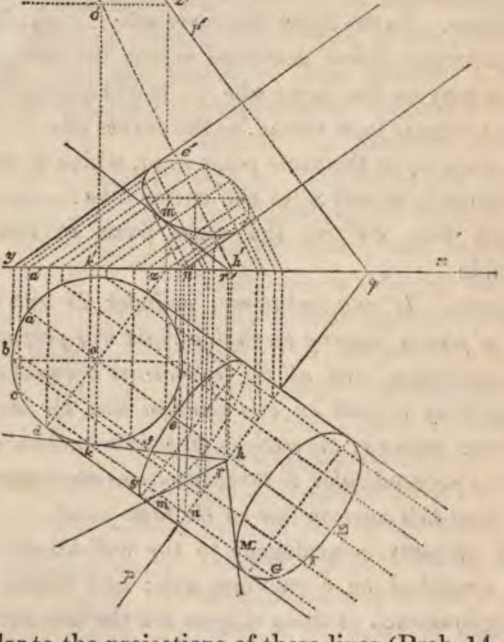
Theorem. If any, whatever, cylinder be cut by any number of planes, passing through a right line perpendicular to its generatrixes, and all the sections are brought into one plane by being rotated on this common line, the tangents to the different curves of the sections, at points in them, situated in one line perpendicular to the common one above-mentioned, will all meet this common line in the same point.

This property is analogous to the well-known one of ellipses described on a common axis; and is true in this case only inasmuch as these ellipses are the intersections of the same cylinder by planes fulfilling the above-stated conditions.

PROBLEM 2.

Section of a cylinder by a plane perpendicular to its generatrixes.

48. Let abc be the horizontal trace of the cylinder, and let the outlines of the cylinder be determined according to the construction to Prob. 1, p. 53. The cutting plane being perpendicular to the generatrixes, its traces must be perpendicular to the projections of these lines (Prob. 14, p. 41); let qp and qp' be these traces. The first step is to find the intersections of the generatrixes with the plane pqp: let a vertical plane be conceived to pass through one of these, $ae\ a'e'$ for example. The horizontal projection of the intersection of this plane with pqp' is ae, and one point in the vertical

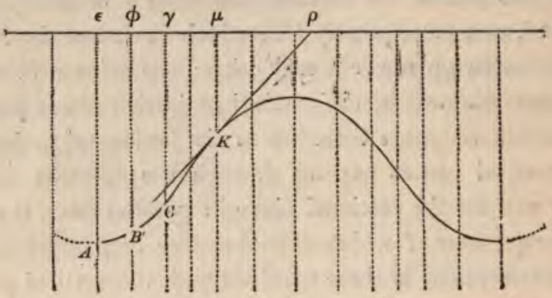


projection is obtained by drawing hh' perpendicular to xy. To determine a second point, draw a parallel, zo, zo' to pq, through any point z,z' of the trace qp'; the point o, in which zo meets ae, is the horizontal projection of a point common to the two planes, the vertical projection o' of this point is deduced, and subsequently $h'o'$, which is that of the line in which the two planes cut each other; the point e' in which ae' meets $h'o'$ will be the vertical projection of one point of the section sought, e therefore is the horizontal projection. The vertical planes passing through the different generatrixes will cut the sectional plane in parallel lines, the vertical projections of which will therefore be parallel to $h'o'$. The intersections of these parallels with the vertical projections of the generatrixes, will determine any requisite number of points in the same manner as e, e' was determined, through which points the section of the cylinder by the plane is to be drawn.

To draw a tangent to this curve at any point m,m', draw kr, a tangent to the horizontal trace of the cylinder, through that of the generatrix which contains the proposed point. This line kr will be the horizontal trace of a plane, tangential to the cylinder. Now the tangent to the section will lie both in this plane, and in the plane of the section, it is therefore their common intersection: hence the point r, in which kr meets pq, is a point in the tangent sought; mr is consequently its horizontal, and $m'r'$ its vertical, projection.

The true figure and magnitude of the section will be found by turning the plane pqp' round on its trace pq, into the horizontal plane; but the constructions requisite will be readily comprehended both from what was explained on this subject in Prob. 1, and from the figure.

When the cylinder is developed, the sectional curve made by the plane qqp', being perpendicular to the generatrixes, will develop into a straight line, perpendicular to the development of the generatrixes. Draw a straight line, and set off on it the lengths of the arcs EF FG... of the section*, from ϵ to ϕ, ϕ to γ, &c.; then make the perpendiculars ϵA, ϕB ... drawn through the points ϵ, ϕ, γ, &c., equal to the real length of the segments of the corresponding generatrixes, intersected between the sectional and the horizontal planes. By this proceeding points in the developed base of the solid may be obtained. The true lengths of these generatrixes are to be deduced by Prob. 5, p. 34.



Let the ordinate Kμ in the development be that which corresponds to mk, $m'k'$ on the cylinder, passing through the point of contact of the tangent; krm, $k'r'm'$ are the projections of a right-angled triangle, of which kr, $k'r'$ is the hypotheneuse, and rM, μK are respectively equal to its two sides; by making $\mu\rho$ equal to Mr, the line joining the points Kρ will be the tangent at K.

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* The generatrixes which pass through the extremities of the equal arcs, ab, bc, cd .... of the base of the cylinder, will not intercept equal arcs of the elliptic section, made by a plane perpendicular to those generatrixes, as will be perceived from the figure E F G ... hence the segments of the straight line $\epsilon\phi$, $\phi\gamma$ ... which are these arcs developed, are not equal, as those in the corresponding construction of the last Problem were.

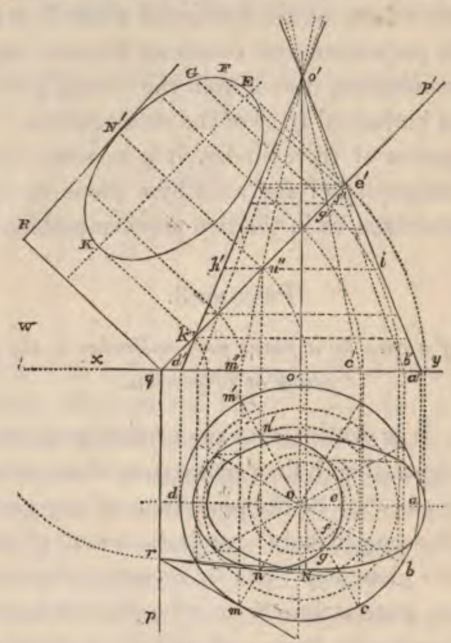
Scholium. When the cutting plane is oblique to the generatrixes, the projections of the curve and of the tangent are found by constructions similar to the foregoing. But to turn the curve down on the horizontal plane, it is necessary to trace the projections and construct the true magnitudes of the perpendiculars, drawn from the various points of this curve to the horizontal trace of the cutting plane. To draw the development of the cylinder, it is necessary previously to obtain the section of the solid by a plane perpendicular to the generatrixes, as in the two previous problems.

PROBLEM 3.

Section of a cone by a plane perpendicular to the vertical plane of projection.

49. 1°. It is evident that the vertical projection of the section is identical with the vertical trace of the cutting-plane pqp'. Let om, $o'm''$ be the projections of any generatrix of the cone; it is easy to find the projections n, n'' of its intersection with the plane pqp'; and by repeating the construction for as many generatrixes as may be deemed necessary, the data for drawing the horizontal projection efg of the conic section may be obtained.

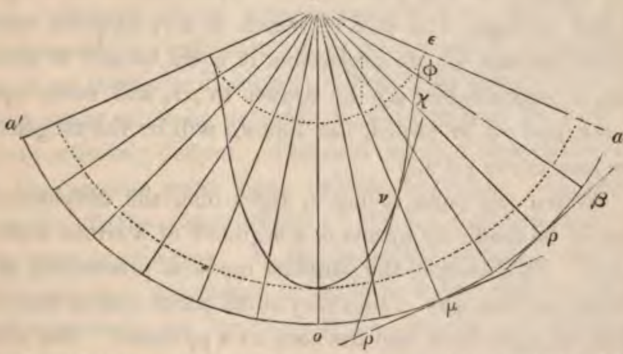
The cone in the figure is a right one with a circular base; its axis is vertical, and the two generatrixes parallel to the vertical plane are oa, $o'a'$, and od, $o'd'$. The points of the curve efg may also be obtained by cutting the cone by horizontal planes: let $h'i'$ be the vertical trace of such a plane, which will obviously cut the cone in a circle, the horizontal projection of which will be an equal circle, described on the centre o, with a radius equal to half $h'i'$; and it cuts the plane pqp' in a line perpendicular to the vertical plane, the horizontal projection nn of which perpendicular is readily determined. The points *n*,*n*, in which this perpendicular cuts the circle, are common to the horizontal projection of the elliptic section, other points in which



may be found by a repetition of the construction made with other horizontal planes. By means of this construction, the points situated on the line $o'o$, perpendicular to xy, may be found, which by the first construction could not readily be done.

2°. To draw a tangent to the curve *efg*, at the point *n*, for example, draw the tangent *mr* to the circular base at the point in which the generator, passing through the proposed point *n*, cuts that base. This tangent *mr* will be the horizontal trace of a plane touching the cone in the generator mo, $m''o'$, and the tangent sought will be the intersection of this tangent plane with the plane pqp'. The point r, in which the trace mr meets the trace pq, will be common to the tangent, or will be its horizontal trace; rn being drawn, will therefore be the projection of that tangent.

3°. To obtain the true form and magnitude of the section the plane pqp' may be turned down either on its vertical trace or on its horizontal trace; in either case the constructions will be precisely the same as those given in the last problem. In the example before us, the major axis of the ellipse is equal to the line $k'e'$, and the minor axis is determined by drawing a horizontal plane through the middle of $e'k'$; the diameter of the circular section made by this plane is equal to the conjugate axis of the elliptic section.



4°. In the case of a right cone, since all the points of the base are at an equal distance from the vertex, this base will develop into a circle, described with oa' for a radius. Let $a\beta\gamma\ldots$ be this circle; make the portion of the circumference $a\ldots\mu\ldots a'$ equal to the periphery of the base of the cone, setting off along it small equal chords, which by trial have been found to divide the circular base into any number of parts.

The radii ωa, $\omega\beta$, $\omega\gamma\ldots$, drawn through these points of division, will represent the corresponding generatrixes oa, ob, $oc\ldots$ of the cone when developed.

The points of the conic section situated on these generatrixes will not change their respective distances from the vertex, and it is evident that these distances are given on the line $o'a'$, between the point o' and the parallels to xy drawn through the projections e', f', g'; hence if these distances are set off from ω to $\epsilon, \phi, \chi\ldots$, the curve $\epsilon\phi\chi$, representing the development of the elliptic section, can be drawn. The tangent is obtained as it was in the case of the cylinder, by observing that before and after the development, each element of the conic section always makes the same angle with the contiguous generatrix. Let the point ν, for example, correspond to the point n, n' of the conic section. The tangent at this point is the hypotheneuse of a right-angled triangle, one side of which is mr, and the other, which has mn for its projection, is equal to $\mu\nu$; if, therefore, a perpendicular $\mu'\rho$ be drawn to $\mu\nu$, and made equal to mr, and $\nu\rho$ be joined, the line $\nu\rho$ will be the tangent to the new curve $\epsilon\phi\gamma\ldots$.

When the cone is not a right one, the development cannot be made by means of a segment of a circle such as $\omega a a'$. In this case, the simplest mode of proceeding is to divide the base $abc\ldots$ into very small parts, and to consider these as right lines, and the cone as a pyramid. The three sides of each triangular face of this pyramid is easily deduced, and they may be constructed successively, and the development of the cone thus obtained. It is equally easy to determine the distances from the vertex of the cone to the different points of the conic section, and consequently to trace the curve produced by the development of this section. The tangent is determined by constructing on $\mu\nu$ an obliqueangled triangle, one side of which, $\mu\rho$, is equal to mr, but the third side $\nu\rho$, must be deduced from its two projections nr and $n''q$.

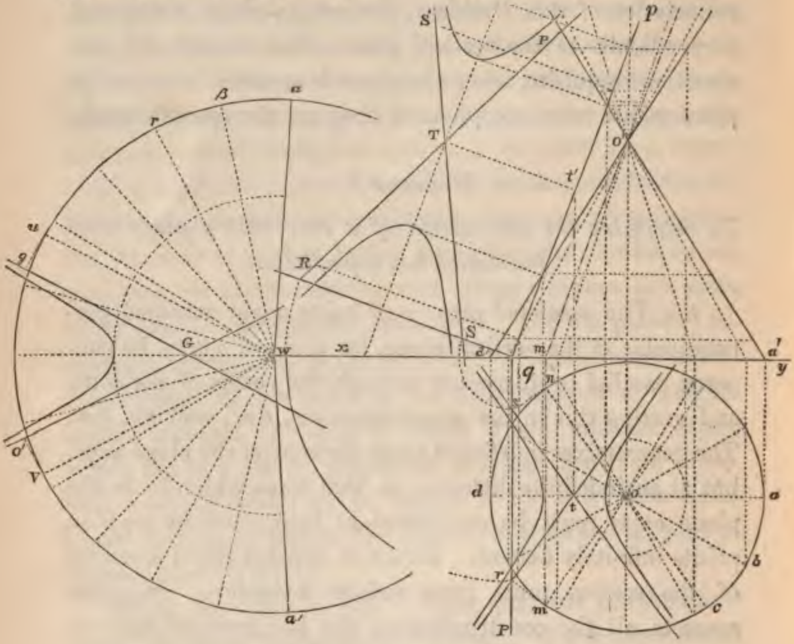
Scholium. It is only for simplicity's sake, that, in the enunciation of this Problem, the cutting-plane is assumed, perpendicular to the vertical plane. But though the constructions requisite, when the plane is assumed in any other position, are more complicated, they are not more difficult.

PROBLEM 4.

To determine the intersection of a cone with a plane when the section has asymptotes.

50. The sectional plane pqp' being again assumed perpendicular to the vertical plane, let a plane, $o'm'm$, be supposed parallel to it, passing through the vertex of the cone, and cutting this in the generatrixes om, $o'm'$, and on, $o'n'$. The generatrixes indefinitely near these meet the plane pqp', but at an indefinite distance; so that those which lie in the plane omm' must be considered as being met by pqp' in points infinitely distant. Hence it appears that the curves of the section must have infinite branches. In other respects, all the constructions of the last problem must be employed in this case. In the figure the assumed cone is a right one, and the section is an hyperbola; the true form and magnitude are determined by turning the sectional plane down on the vertical plane; the one conical surface is alone developed in the figure.

The principal object in this problem is to determine the asymptotes to the hyperbolas, or, in other terms, the tangents at points in the curves infinitely distant, which points may be considered as lying in the generatrixes parallel to the plane pqp'. Now, if we conceive a tangent to be drawn to the curve anywhere, and the point of contact to be more and more distant, the limit to this tangent will be an asymptote, consequently these asymptotes may be determined on precisely the same principles as any other tangents.



The tangent at any point of the conic section is the intersection with the plane pqp' of a plane tangential to the conical surface, touching it in the generatrix containing the point. Draw the tangents mr, ns to the base $abc\ldots$; these will be the horizontal traces of the tangent planes, touching the cone in the generatrixes passing through the infinitely distant points of the curve; hence the points r and s, in which these traces meet pq, are common to the asymptotes. If, therefore, through r and s, parallels to the generatrixes om, on be drawn, they will be the horizontal projections of the asymptotes. These asymptotes to the curve, when drawn of its true magnitude and figure, as well as to the curve of the developed section, are easily determined from the foregoing by principles explained in the preceding problems.

§ 2. INTERSECTIONS OF CURVED SURFACES.

PROBLEM 1.

Intersection of two cylinders.

51. The auxiliary planes may be assumed parallel to the generatrixes of the two solids, in which case these can only be cut by the planes in lines coinciding with the generatrixes; hence the points of the intersection sought are determined by means of right lines only.

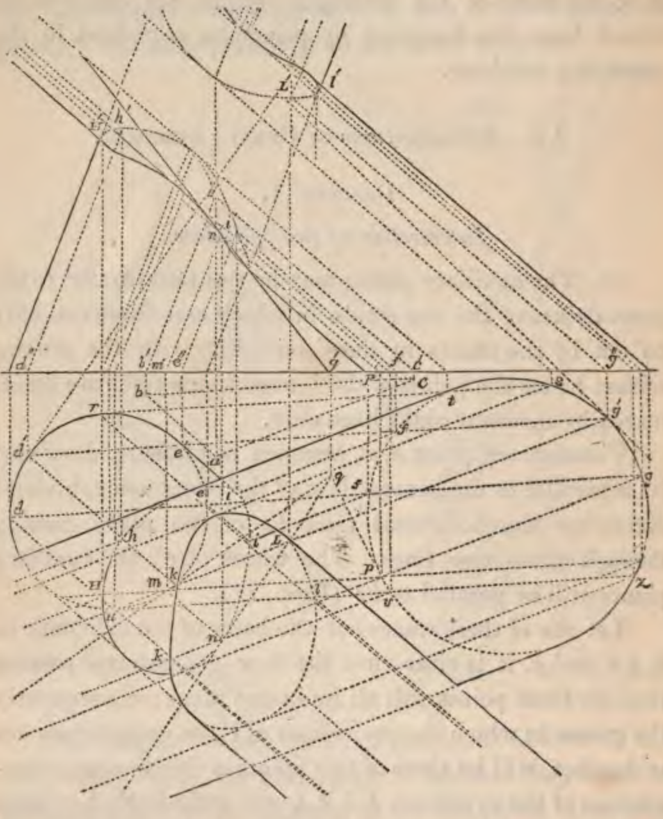
Through any point a, a', assumed at pleasure, draw two lines parallel to the generatrixes of the two given cylinders: determine the horizontal trace bc of the plane passing through these two lines. The traces of all the auxiliary planes will be parallel to bc (Prop. 17).

Let one of these traces cut the bases of the cylinders in d, e, f, and g, it is clear that the four generatrixes passing through these points will all lie in one plane; consequently the points in which the projections of these generatrixes cut each other, will be those of two points in the common intersections of the cylinders; h, i, k, l, will accordingly be points in the horizontal projection of this curve, and $h'i'k'l'$ their vertical projections.

If hh', ii', kk', ll' be joined, these four lines will be perpendicular to the ground line.

By a repetition of these constructions as many points in the curve of intersection may be obtained as may be deemed necessary.

The tangent at any point of the curve of intersection will be that of two planes, one tangential to each cylinder in the proposed point. Let the point n, n', for example, be



the intersection of two generatrixes, cutting the bases of the cylinders in m, m' and p, p'. Draw mq, pq tangents to these bases at m and p. These tangents will be the horizontal traces of the planes tangential to the cylinders in the two generatrixes through n; the line nq, therefore, drawn through n and q, the point common to these two traces, will be the horizontal projection of the intersection of the tangential planes, or will be that of the tangent sought; q' in *xy* will be the vertical projection of *q*, and *n′q′* therefore that of the same tangent.

Of two cylinders which intersect, one may either entirely pass through the other, in which case there will be two curves of intersection, or a part only of the generatrixes of the one may meet the other cylindrical surface; when only one curve will be formed. The following construction will determine at once whether two given cylinders will intersect each other according to one or the other of these conditions.

Draw two tangents to the base of each cylinder parallel to *bc*. If both the tangents to either base cut the other, as *rs*, *uv*, the auxiliary planes, of which these lines are the traces, comprise between them the whole of the one solid, but only the portions of the second corresponding to the arcs *sgz*, *tfv*, and it is obvious that the first cylinder meets these two portions of surface, and traverses the space between them; in this case, therefore, there will be two curves of intersection.

But if each base be cut by one only of the tangents drawn to the other as is the case in the next fig.*; then the auxiliary planes passing through the tangents *rs* and *uz* comprise the portions of surface of both cylinders corresponding to the arcs *urv* and *szt*, and it is evident that these portions mutually meet; while the remaining or the exterior portions have no common point; in this case there will therefore be only one curve of intersection.

The constructions above described for determining whether the cylinders meet in two, or only in one continuous curve determine the generatrixes of each solid, which serve as limits to the curve of intersection. Thus in the first fig., the curve in which the one cylinder *enters* the other is bounded by the generatrixes which pass through the points *t* and *v*, and the other curve by which the cylinder passes out is bounded by the generatrixes which pass through *s* and *z*.

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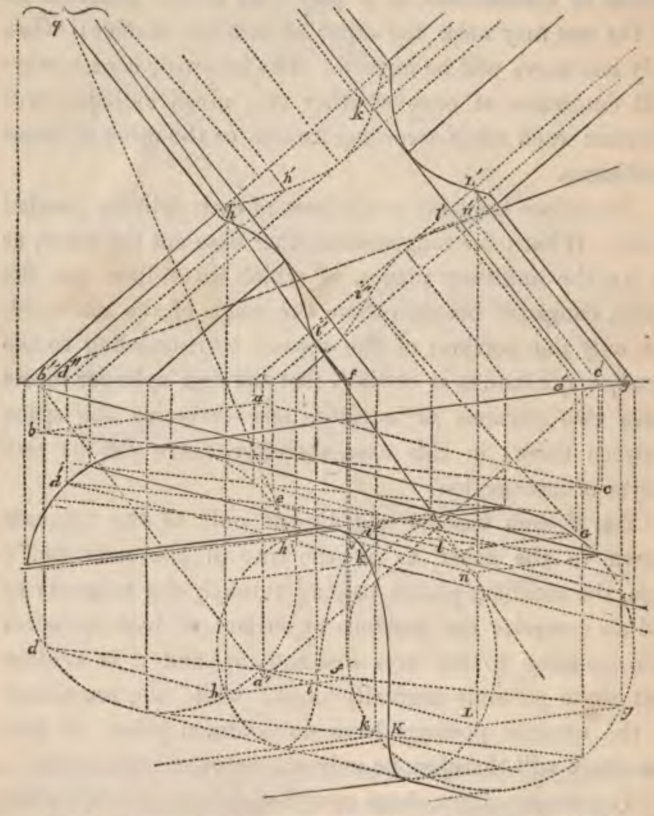
* Refer to the former fig. for the letters of reference not inserted in the second.



In the above fig. the curve of intersection has for its limits on one of the cylinders the generatrixes through *u* and *v*, and on the other those which pass through *s* and *t*.

The general construction for determining the tangent to these curves, also proves that these generatrixes are tangents to the curve of intersection; thus for example, let one of the planes, tangential to one of the two cylinders, pass through the point *r* and the other through *t*. The former coincides with the auxiliary plane, the trace of which is *rt;* it accordingly intersects the second in the generatrix which passes through *t;* this therefore is a tangent to the curve of intersection. The projections of this generatrix are consequently tangents to that of the curve.

PROBLEM 2.

Intersection of two Cones.

52. A series of planes passing through the two vertices of the cones will cut their surfaces on straight lines, the intersections of which will be points in the curve sought. The tangent at any point of the curve will be the intersection of the planes tangential to the two cones.

Let the given cones be those represented in the figure, let that with the least base be called, for distinction, the "small," and the other the "great cone." Let *c* be the horizontal trace of the line *ab*, *a′b′* joining the vertices. The horizontal traces of all the auxiliary planes will pass through this point *c*. Draw the tangent *cd*, *ce* to the base of the small cone, which will cut that of the larger in *f*, *g*, *h*, and *i*. All the auxiliary planes, the horizontal traces of which as *cm* are comprised between these tangents, cut both cones and will determine points common to their surfaces; but those planes, of which the traces are external to these two tangents, cannot cut both surfaces.

Hence the intersection of these cones is composed of two parts*, one situated on that portion of the larger cone corresponding to the arc *frmg*, and the other on that corresponding to the arc *hi*, but in the figure the constructions appertaining to the former are alone shown.

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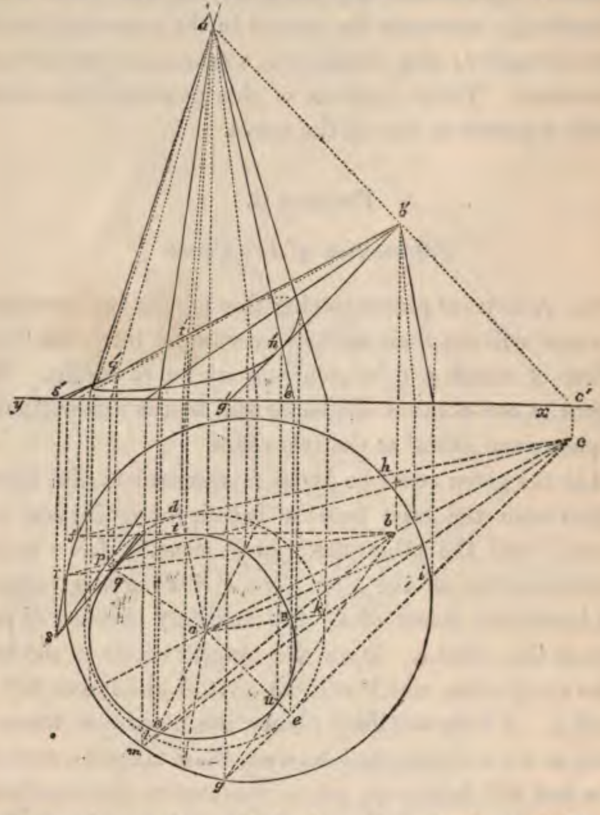
* The conical surface, it must be remembered, consists of two parts, one on each side of the vertex; only one of these is shown in the figure.



The points *a*, *b* are the horizontal projections of the vertexes, the lines *ad* and *bf*, *ae* and *bg*, are those of the generatrixes in which the cones are cut by the auxiliary planes which have *cf*, *cg*, for their horizontal traces, consequently the points *t*, *u* in which these respectively meet, will be the horizontal projections of two points in the curve of intersection. It must be observed that this curve being bounded on the larger cone by the generatrixes projected in *bf*, *bg*, will have these line for tangents, and consequently the projection of that curve will have *bf*, *bg* for tangents.

Let *cm* be the trace of any auxiliary plane and let it cut the bases in *k*, *l*, and *m;* this plane will cut the conical surfaces in generatrixes having *ak*, *al*, *bm* for their projections. The intersections *n*, *o* of *bm* with *ak*, *al* will be points in the horizontal projection of the intersection of the surfaces, and by a repetition of these constructions as many points may be obtained as may be deemed necessary.

The other projection is obtained by constructing the vertical projections of the generatrixes determined by the auxiliary planes, and will be readily understood from the figure without further description.

Let it be required to determine the tangent to the curve of intersection at the point q, q' that in which the generatrixes *aq*, *br* meet. Draw the tangents to the base *ps*, *rs*, which will obviously be the horizontal traces of planes tangential to the conical surfaces in the generatrixes in question; draw *qs* through the point *s* in which the tangents meet; *sq*, $s'q'$ will be the horizontal and vertical projections of the tangent at the point qq'.

If this construction were applied to the points t, t', u, u', the lines *bt*, $b't'$, *bu*, $b'u'$ will be obtained, as, obviously, ought to be the case.

Problem 3.

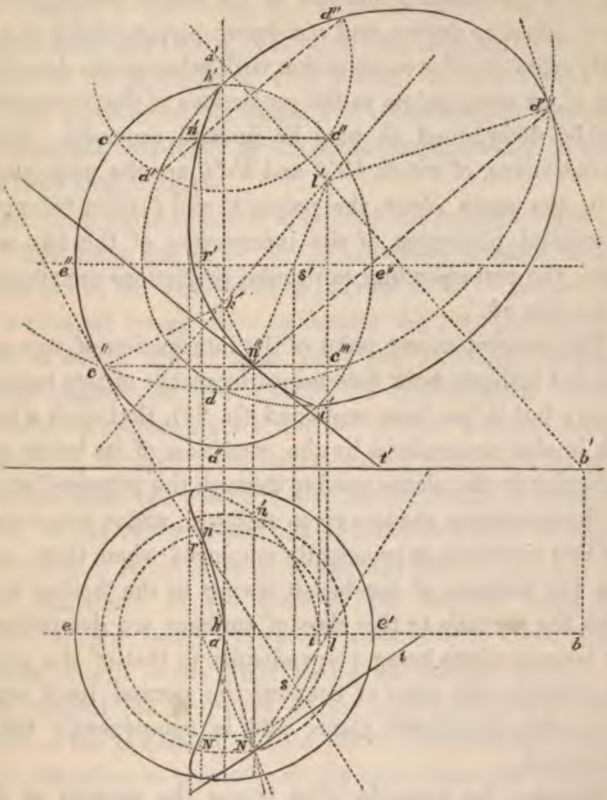
To determine the intersection of two surfaces of revolution, the axes of which meet each other.

53. The construction would be simple enough in the case of two surfaces, the axes of which were parallel; for in that

case we should only have to draw a series of planes perpendicular to these axes, each cutting the surfaces therefore in circles, the points of intersection of which must belong to the curve sought. But this method of proceeding is not applicable when the axes are not parallel; for then the planes perpendicular to one will not be so to the other. Nevertheless, if the two axes meet, as by the hypothesis they are supposed to do, the points of the curve may be determined by the intersections of circles; the surfaces being cut, not by planes but by concentric spheres, having the point of intersection of the axes for their common centre. For each of these spheres will intersect the two surfaces in the circumferences of circles, the intersections of which will be points in the common section of those surfaces.

To render the constructions as simple as possible, let the vertical plane be assumed parallel to the two axes, and the horizontal plane perpendicular to one of them. On this supposition, one of the axes will have a point a, for one of its projections, and the other a line ab, parallel to xy. The vertical projection of the first axis will be $a'a''$, perpendicular to xy, and the other may have any direction, as $a'b'$, according to the data. The two surfaces are given by the vertical projections of two *meridians*, $h'c'i'c''$ and $h'd'i'd''$, in the plane of the axes.

From a' describe any circle at pleasure, $c'd'c''d''$ cutting the projections of the two meridians in c', c'', d' and d''. This circle will obviously be the projection of the section of a sphere of the same radius, having a, a' for its centre, made by the plane of the two axes. The points $c'c''$, $d'd''$, are therefore the vertical projections of points common to these meridians and to the sphere. If the meridian of the first surface be supposed to rotate on its axis, the points $c'c''$ would describe a horizontal circle, the vertical projection of which would be $c'c''$; this circle would be common to the surface and to the sphere.



For the same reasons, $d'd''$ is the projection of a circle common to the sphere and to the second surface; the point n' in which the lines $c'c''$, $d'd''$ meet, being in the interior of the circle $c'd'c''d''$, it follows that the perpendicular to the vertical plane through this point, cuts the spherical surface in two points, belonging to the two circles in question, and which therefore lie in the common intersection of the given surfaces; these points obviously have only one common vertical projection. To determine their horizontal ones, N, n, the horizontal projection of the circle, corresponding to $c'c''$, must be drawn, and $n'n$ drawn perpendicular to xy.

By repeating this construction with other circles described from a', as many points in the projections of the intersection may be determined as may be deemed requisite. Since the meridians, of which $h'e'i'$ and $h'd'i'$ are the projections, lie in the same plane, the points h' and i' must belong to the vertical projection of the intersection of the two surfaces; the corresponding horizontal projections are situated on the line ab.

The tangent at any point of the intersection of two surfaces has hitherto been determined by means of two tangent planes; but it has been explained (p. 64), that such a tangent is also determined by the condition of its being perpendicular to the planes passing through the proposed point, and the normals to the two given surfaces; and it was stated, that this condition is principally employed when those surfaces are surfaces of revolution, owing to the facility with which the normals to this class of surfaces are determined. The tangent plane being perpendicular to that of the meridian, through the point of contact, the normal itself must lie in this meridional plane, and is consequently easily determined.

Suppose, for example, that it was the tangent at the point N$'$, n'', which was to be drawn, this point lying on the two circles projected in $c'c'''$, $d'd'''$; the two normals at that point must first be obtained. To effect this it must be observed, that the normals drawn to any surface of revolution through the different points of a *parallel*, must all meet in one point of the axis. If, therefore, the normals $c''k'$ and $d'''l'$, to the vertical projections of the two meridians are drawn, cutting the axis in k' and l', and $n'k'$, $n'l'$ be joined, these lines will be the vertical projections of the two normals sought; a and l are the two horizontal projections corresponding to k' and l', $\mathrm{N}a$, $\mathrm{N}l$, are consequently the projections of the normals on that plane. It would next be necessary to construct the traces of the plane passing through these normals, and lastly to draw the perpendicular to this plane from the point nn'.

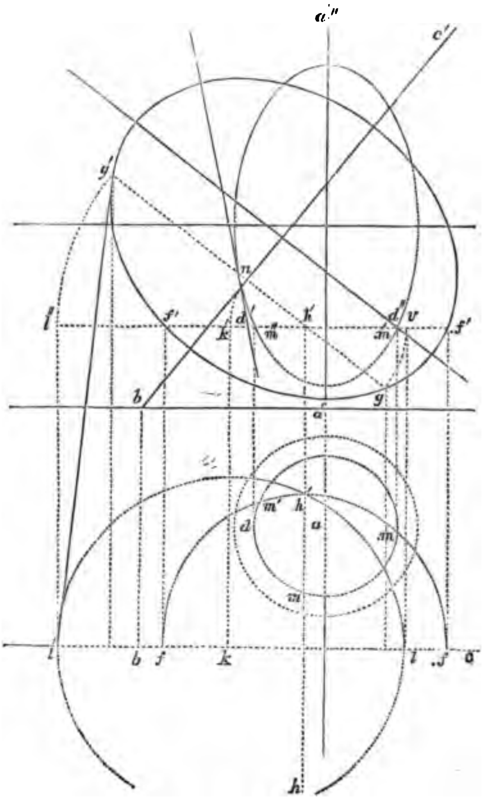
But it is unnecessary to construct the vertical trace of this plane, since the plane of the axes being parallel to the vertical plane, this trace will be parallel to $k'l'$. Instead of the horizontal trace, we may employ the line joining the points in which the normals meet any horizontal plane; that, for example, of which $e'e''$ is the vertical trace. Consequently, by prolonging $n''k'$ and $n''l'$ to their points of intersection r', s', with $e'e''$, the corresponding points r, s, on the horizontal plane can be deduced on the horizontal projections $\mathrm{N}a$, $\mathrm{N}l$, of the two normals, and rs must be drawn through these points. After having found the lines rs and $k'l'$, to which the traces of the plane of the normals must be parallel, nt, $n't'$ being drawn perpendicular to them from the point n, will be the projections of the tangent at $\mathrm{N}'n''$.

In general the projection of the tangent to a curve is a tangent to the projection of that curve; nevertheless there is one exception to this rule, in the case when the tangent is perpendicular to the plane of projection: for then the projection of the tangent is a point. This would happen with the tangents at the points projected in h' and i', for the tangent planes to the two surfaces at these points would evidently be perpendicular to the vertical plane. Hence if the tangent were to be determined by the intersection of the tangent planes, the construction would fail in this case as far as the vertical projection of that tangent is concerned, and recourse must then be had to the normals as above described.

Problem 4.

To determine the intersection of two surfaces of revolution, the axes of which lie in different planes.

54. The horizontal plane must be again assumed perpendicular to the axis of the first surface, and the vertical plane parallel to the axes of both; the data will present the same



arrangement as in the preceding problem; only the horizontal projection of the axis of the second surface will not pass through the point a, but will be parallel to xy, as bc.

Since the axes lie in different planes, there can be no two *parallels*, one on each surface, which lie in one plane, or are comprised in one sphere; another proceeding must therefore be adopted,—that of cutting the two surfaces by horizontal planes. Each of these will intersect the first in a circle, the horizontal projection of which will be an equal circle, which is readily described; but the section of the second surface will be a curve, the projection of which must be obtained by means of points, through which this curve is drawn; the points in which it cuts the circular section of the first surface will be points in the projection of the common insersection sought; and the vertical projections of these points will lie in the trace of the auxiliary plane. Every two points thus to be determined will necessitate the construction of a curve.

Let $d'f'$, parallel to xy, be the vertical trace of an auxiliary horizontal plane; its intersection with the first surface is projected vertically in $d'd''$, and horizontally in the circle dmm, described from a as a centre, with $\frac{1}{2}d'd''$ for a radius; the vertical projection of the curve of the section with the second surface will be $f'f''$; and to obtain points in the horizontal projection, the surface must be supposed cut by planes perpendicular to its axis, forming consequently circular sections.

Let $g'g''$ be the vertical projection of such a circular section, the point h', in which $f'f''$ meets $g'g''$, is the vertical projection of two points in the auxiliary curve required; consequently the horizontal projections of these two points must lie in $h'h$, drawn perpendicular to xy. Let the plane of the circle $g'g''$, be supposed turned round on the line $h'h$ till it is horizontal, and then projected on the horizontal plane; it will be seen that this projection must be a circle, described from the point k, with kl for its radius. Now, the points projected in h', before mentioned, are not moved on this supposition; their horizontal projections, therefore, will be hh, in which the circle kl cuts $h'h$.

By means of other planes perpendicular to $b'c'$, other points of the section corresponding to $f'f''$ are determined; and the curve itself hfh obtained. This curve and the circle dmm, are then the horizontal projections of the sections of the two given surfaces made by the auxiliary plane $d'f'$; and hence m, m', m', m'', in which these lines cut each other, must be two points of the intersection of those surfaces. These constructions must be repeated for other auxiliary planes, till a sufficient number of points are determined*.

The tangent to the intersection of the two surfaces is obtained, as before, by means of normals. The vertical projections of the *parallels* of each surface in which the point of contact is situated, must first be determined, and subsequently those of the normals; but instead of finding the traces of the plane of the normals, which would be an unnecessary operation, the intersections of this plane with the horizontal and vertical planes, passing through the point in which the normal to the second surface meets its axis, are determined.

Problem 5.

To construct the projection of a helix on a vertical cylinder, and to determine the tangents to this helix, which are parallel to a given plane.

55. Any whatever cylinder being given, let it be supposed cut by a plane perpendicular to its generatrixes; if distances

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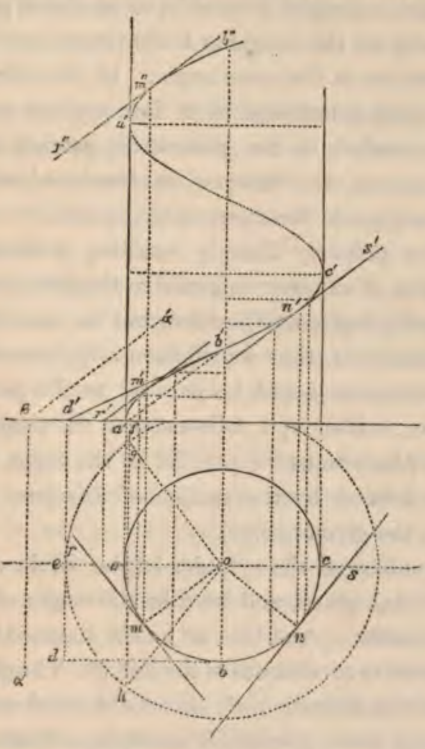
* To avoid complicating the figure, the constructions for finding the points, m, m', m', m'', are alone shown, and not the intersection of the two surfaces.

are set off along these generatrixes from the plane proportional to the arcs of the base comprised between these generatrixes, and any fixed point of this base, assumed at



pleasure, the curve which would be thus determined is a helix*.

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* If a right-angled triangle, the base of which is equal to the circular base of a right cylinder rectified, and its height any whatever, be conceived as wound round the cylinder, so as to coincide everywhere with its surface, the hypotheneuse will form a helix on the cylinder. The side of the triangle will coincide with the directrix passing through the point of departure of the curve.

This mode of generation shows that if the cylinder be developed on a plane, the base becomes a right line, perpendicular to the generatrixes; and the helix, consequently, develops into a straight line, oblique to these generatrixes. Consequently all the tangents to the curve are inclined to the generatrixes in the same angle. If, therefore, we develop the cylinder and the helix into a plane, touching the cylindrical surface in the generatrix, passing through the point of contact, the helix when developed will coincide with the tangent to the curve at that point.

Another property directly resulting from these conditions is, that if all the tangents to the helix were moved parallel to their original position, but so as to pass through one common point, they would form a right conical surface, the axis of which would be parallel to the generatrixes of the cylinder. Hence it follows that the tangents parallel to a given plane must be parallel to the right lines, which would be formed by the section of this cone by a plane parallel to the given one.

Let the base of the cylinder be the circle abc lying in the horizontal plane, and let a be the origin of the arcs or circular abscissa. And let $a'a''$ be the distance between any two consecutive revolutions of the helix*. The circumference $abca$, and the altitude $a'a''$, must be divided into any number of equal parts, twelve for example. Draw the vertical projections of the generatrixes passing through the points of division of the base, and make the heights of these projections in succession equal to the corresponding segments of $a'a''$, by drawing lines parallel to xy through the points of division of that line. The line $a'b'c'a''$ drawn through these points will be the vertical projection of one convolution of the helix. And any number of consecutive revolutions may be drawn by setting off the constant distance $a'a''$ along the projections of the generatrixes, from each point of the first, which is obviously equivalent to a repetition of the construction by which the first convolution was obtained.

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* This distance $a'a''$ will therefore be the altitude of the right-angled triangle alluded to in the preceding note.

Let it now be required to determine the tangents to the helix parallel to any given plane, $a\beta a'$, the horizontal trace of which may be assumed perpendicular to xy; but the construction will be equally simple if the plane have any proposed position.

Draw the tangent bd to the circle abc, and make it equal to the arc ab rectified. The vertical plane passing through bd would be tangential to the cylinder in the generatrix $b\,b'b''$, and the helix would develop on this plane into the tangent to the curve at b, b', which would pass through the point d, d'. $b'd'$, therefore, drawn through d', the vertical projection of d, will be that of the tangent sought, and bd will be its horizontal projection.

A line parallel to this tangent through the point o, b', would have e for its horizontal trace, and if this line were rotated round the axis of the cylinder, it would generate a right cone, having a circular base of the radius oe, the generatrixes of which cone are, as has been stated, parallel to the tangents of the helix: consequently the plane $b'fh$, which passes through the vertex of this cone, and which is parallel to the given plane $a'\beta a$, must cut the cone in lines parallel to the required tangents; these lines have og, oh, for their horizontal projections.

And since the tangents to the helix have tangents to the circle abc, for their horizontal projections, if the tangents mr, ns, be drawn respectively parallel to og, oh, they will be the horizontal projections of the tangents sought.

The vertical projections m', n' of the proposed points being determined by means of the horizontal ones, $m'r'$, $n's'$ are to be drawn parallel to βa or bf, for the vertical projections of the tangents to the helix.

Although two other tangents to the circle abc might be drawn, parallel to og, oh, respectively, yet they would not be the horizontal projections of two tangents to the helix parallel to the generatrixes of the cone; it being obvious, from the nature of the curve, that the tangents to it at those points would be inclined in the opposite direction to that of the corresponding generatrix.

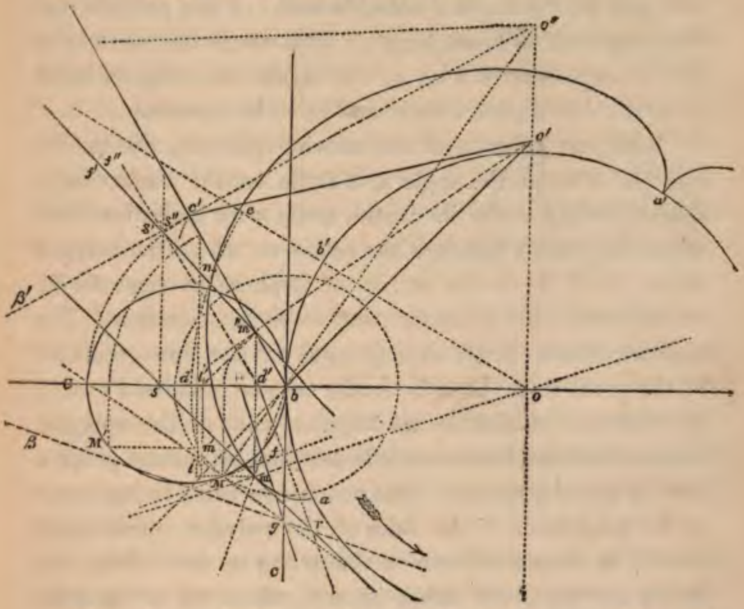
There would, however, be two tangents parallel to the two first at each successive spire of the helix, at points having m for their common horizontal projections.

Problem 6.

To construct the projection of a spherical epicycloid, and to determine a tangent to it at any proposed point.

56. Def. If a circle roll on the circumference of another, which remains stationary, so that the two circumferences are always tangential to each other, any point in that of the first will describe a curve termed an epicycloid: if the moving circles always remain in the same plane with the fixed one, it is clear that the epicycloid will be a plane curve lying in the same common plane. But if the two circles are not in the same plane, but constitute the bases of two right cones having a common vertex, and therefore equal generatrixes, and the one cone roll over the surface of the other fixed one, so that they may always have the same common vertex, and touch in a line common to both, then the epicycloid is termed a spherical one.

For, on this supposition, it is clear that the fixed circle is on a sphere, having the common vertex of the cones for its centre, and their common generatrix for its radius; and the moving circle always remains on this spherical surface, or, in every position, lies entirely in it.



The plane of this moving circle, in every position of it, cuts that of the fixed base in a line which is a tangent to both circles, and it is obvious that the inclination of the two planes remains constant, and equal to the angle formed by the axes of the two cones.

The tangent at any point of the spherical epicycloid must be in the tangent plane to the sphere, and it will now be proved that it also lies in a plane tangential to a second sphere, which would have for its centre the point in which the moving circle touches the fixed one in that position.

Instead of circles, let two polygons* ABC . ., A′B′C′ . ., be supposed to roll one on the other; and to simplify the subject, let the polygons be supposed regular ones of any number of sides at pleasure, the sides of the one being equal to those of the other, although the number of them may vary in the two, and let the angle A′ coincide with A; the polygon ABC being assumed as fixed, let A′B′C′ turn round the point A, so that B′ may coincide with B: then again the polygon turns on B, till C′ coincides with C, and so on in succession.

Now, any point M of the moving polygon, during the rotation of it on the angle A, will lie on the surface of a sphere having A for its centre, and AM for its radius; and when the point B becomes the centre on which the polygon turns, M will lie on the surface of a sphere having B for its centre and BM for its radius, and so on in succession. The tangent, consequently, at any point of the line generated by the point M, ought to lie in the tangent plane to a sphere, the centre of which is the angular point of the polygon, about which the rotation is effected that brings the point M into its actual position. This conclusion being independent of the magnitude of the sides of the polygon, must apply equally in the case of two curves rolling on each other, that is, the moving curve being in any, whatever, of its positions, and if H express the point in which it touches the fixed curve, and N the corresponding situation of the generating point M: the tangent at the point N of the curve generated by the point M, will always be contained in the tangent plane at the point N to the sphere having H for its centre and HN for its radius.

1°. *To draw the projection of the Epicycloid.* This projection may be made on the plane of the base of the fixed cone; let the circle abb, a, described on the horizontal plane with the radius oa, be the base of this cone, and let the point which generates the epicycloid have originally been at a, and the point of contact of the two circles be supposed now at b; to determine the actual position of the generating point a second plane of projection must be assumed cutting the first in ob, it must consequently pass through the common vertex o, o' of the two cones, and will cut the moving cone in the isosceles triangle $o'bc'$, which is one of the data of the problem; the horizontal trace of the plane of the base of this cone will be the line bc, a tangent to the fixed circle in the point b, while bc' is obviously the vertical trace, and at the same time the vertical projection of the base of the cone.

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* To avoid unnecessary complexity, the polygons are not drawn in the figure.

The circle bMC, the diameter bC of which is equal to that of the base of the moving cone bc', will be that base turned down on the horizontal plane. Make the arc bM of this circle equal to the arc ba, for since the point M was originally at a, and the different elements of the two arcs bM and ba, having been successively applied to each other, these two arcs must be equal.

Make bm' in bc' equal to the distance of M from bc; through m' draw a line parallel, and through M one perpendicular to bc; these two lines will cut each other in m, the projection of the point M on the horizontal plane, and therefore a point in the projection of the epicycloid.

By a repetition of this construction, as many points in the projection as may be deemed necessary may be determined, but instead of repeating the construction at different points of the circle $abb'a'$, as for example at b, all that part of the construction which requires the circle bMC, may be made on that circle; thus, the arc bM is to be made equal to ba, and the horizontal projection m, of the point

M, found as before, and then *m*, must be transferred to its true position by taking the point *n* in the same position as regards *ob*, that *m* has to *ob*. Which is done by describing a circle from *a* as a centre with *om*, for a radius, and making the arc *en* equal to *dm*, of that circle.

2°. *Construction of the Tangent.* The epicycloid, as has been already observed, lies on the surface of a sphere, having *o*, *o'* for its centre, and *o'b* for its radius, consequently the tangent to the point *m*, *m'* of this curve is in the tangent plane to this sphere. It has also been proved to lie in the tangent plane to a sphere, the centre of which would be in *b*, and the radius of which would be *b*M; the intersection of these two tangent planes must therefore be the tangent in question.

The base of the moving cone, which is turned down in the circle *b*MC, belongs to the first of these spheres; if therefore a tangent M*f* is drawn to the circle *b*MC at the point M, it will be a line situated in the tangent plane to the sphere turned down with *b*MC on the horizontal plane, consequently the point *f*, in which M*f* meets *bc*, is in the horizontal trace of this plane. This trace must therefore be the line *d'f* drawn from *f*, perpendicular to the projection *om* of the line, which passes through the point of contact.

The circle described with the radius *b*M is one of the second sphere, also turned down on the horizontal plane, if M*g* be drawn a tangent to it, meeting *bc* in *g*, and $g\beta$ be drawn perpendicular to *bm*, the line $g\beta$ will be the horizontal trace of the tangent plane to the second sphere. The traces *d'f*, βg meet in *r*, which must belong to the intersection of the two tangent planes, the line *mr*, drawn through this point *r* and through *m*, must consequently be the *projection* of the tangent to the epicycloid, or, which is the same thing, a tangent to the projection of that curve.

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